STABILITY OF ISOPERIMETRIC INEQUALITIES

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ABSTRACT. We review stability results for the classical and anisotropic isoperimetric inequalities as present in [Maggi, "Some methods for studying stability in isoperimetric type problems", Bull. Amer. Math. Soc.] and [Figalli, Maggi and Pratelli, "A mass transportation approach to quantitative isoperimetric inequalities", Invent. Math.], using symmetrization techiques and Gromov's proof of the isoperimetric inequality.

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July 2022

1 Introduction

The classical isoperimetric problem is a topic of very rich history and interest. It states that *among all shapes with equal volume, the ball is the one with the smallest perimeter*. This variational principle is the answer to a natural question dating back to the times of Ancient Greece, and closely related to Dido's problem about the foundation of the city of Carthage.

The analytical expression of this classical problem is the isoperimetric inequality, which states that for a measurable set $E \subset \mathbb{R}^n$, $n \geq 1$ with Lebesgue measure $|E|$,

$$n\,|B|^{1/n}\,|E|^{(n-1)/n} \leq P(E), \quad \text{for } |E| < \infty.$$

Here B is the unit ball in $\mathbb{R}^n$ and $P(E)$ is the *distributional perimeter* of E, which can be thought of as the classical (or Euclidean) perimeter of E. This defines a functional on the measurable sets of $\mathbb{R}^n$, which is *translation invariant*, agrees with the *elementary notion of perimeter on sets of smooth or polyhedral boundary*, and obeys the *scaling law* $P(\lambda E) = \lambda^{n-1} P(E)$. These properties imply that if $B(x,r)$ is a ball of equal measure to E, then $P\left(B(x,r)\right) \leq P(E)$.

Apart from being a topic of intrisic interest, the isoperimetric inequality and its variants appear very frequently in physics and geometry. They are connected to the shape of planets and orbits, and they appear in the characterization of isometries and stability of minimal surfaces [Os2, pp. 1218-1222]. A large number of practical applications can be found in the book of Pólya and Szegö [PS].

A particularly interesting generalization of the isoperimetric inequality is the *anisotropic isoperimetric inequality*, which arises in relation to a natural generalization of the Euclidean perimeter. In this, an open, bounded, convex set $K \subset \mathbb{R}^n$ containing the origin is used to weigh the directions in $\mathbb{R}^n$ by means of the scalar product. The notion that originates with these new weights is the *anisotropic perimeter* P_K. It is key when modelling surface tension in the study of equilibrium configurations of solid crystals [Wu, He, Ta], and constitutes the basic model for surface energies in phase transitions [Gu]. In particular, the anisotropic isoperimetric inequality states that

$$n\,|K|^{1/n}\,|E|^{(n-1)/n} \leq P_K(E), \quad \text{for } |E| < \infty.$$

The aim of this paper is to review and explain the approaches and methods described in both [Ma2] (a survey of [FMP]) and [FiMP], introducing and proving the results present in both papers that show stability of both the Euclidean and the anisotropic isoperimetric inequalities, as well as the inequalities themselves. In modern mathematics, the natural framework for studying isoperimetric problems is the theory of sets of finite perimeter as developed by De Giorgi. While it is true that many techniques and methods used throughout the paper can be understood heuristically without introducing the theory, rigorous justification of the arguments used when proving the main results requires some terminology and basic facts. To this end, section 1.1 will be

devoted to explaining, or at least giving the intuition regarding the required results. In doing so we will follow chapters 12, 13, 15 and 16 of the book by Maggi [Ma1]. After that, we will prove the isoperimetric inequality in part 2, followed by a stability result for it in part 3. The last two parts will be devoted to the discussion of the anisotropic isoperimetric inequality and its stability, and we shall remark the key differences between the approaches in [Ma2] and [FiMP] for proving stability in both settings at the beginning of part 5.

1.1 Sets of finite perimeter

When trying to define the perimeter of a measurable set $E \subset \mathbb{R}^n$, it is not enough to simply set $P(E)$ to be $\mathcal{H}^{n-1}(\partial E)$. For sets with smooth boundary this definition works, but it fails as soon as E has a more complicated boundary. Indeed, if $E = B \cup \mathbb{Q}^2$ in $\mathbb{R}^2$, then the perimeter of E should be that of B, namely 2π, but instead $\mathcal{H}^{n-1}(\partial E) = \infty$. This suggests that we need a different approach towards defining $P(E)$.

1.1.1 Perimeter and relative perimeter

The starting point of the theory of sets of finite perimeter is a generalization of the Gauss-Green theorem based on the notion of *vector-valued Radon measures*. A $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$ is simply a bounded linear functional μ on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$, and we use the notation

$$\langle \mu, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \cdot \mathrm{d}\mu$$

to indicate the value of μ at $\varphi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m)$. This is of course motivated by the fact that any such functional L induces a $\mathbb{R}^m$-valued set function on bounded Borel sets via Riesz's theorem [Ma1, Theorem 4.7],

$$\begin{array}{ccc} \mathcal{B}_b(\mathbb{R}^n) & \longrightarrow & \mathbb{R}^m \\ E & \longmapsto & \int_E g \,\mathrm{d}|L| \end{array}$$

which is σ-additive on every disjoint sequence of Borel subsets of a compact set in $\mathbb{R}^n$.

Definition 1.1 (Set of finite perimeter)**.** We say that a measurable set $E \subset \mathbb{R}^n$ is of *finite perimeter* if

$$\sup\left\{ \int_E \operatorname{div} T(x)\,\mathrm{d}x : \ T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \sup_{\mathbb{R}^n} |T| \le 1 \right\} < \infty. \tag{1.1}$$

It is easily shown using Riesz's theorem and the density of $C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ in $C_c^0(\mathbb{R}^n;\mathbb{R}^n)$ [Ma1, Proposition 12.1] that if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ such that

$$\int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot \mathrm{d}\mu_E, \quad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n) \tag{1.2}$$

with $|\mu_E|(\mathbb{R}^n) < \infty$, then E is of finite perimeter, and conversely, E is of finite perimeter if there exists such a measure satisfying (1.2). We call μ_E the *Gauss-Green measure* of E, and the perimeter of E is the total variation of μ_E,

$$P(E) = |\mu_E|(\mathbb{R}^n).$$

Of course (1.2) is equivalent to

$$\int_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu_E, \quad \forall \varphi \in C_c^1(\mathbb{R}^n), \tag{1.3}$$

and this expression can be stated as the distributional derivative $D\mathbb{1}_E$ of $\mathbb{1}_E$ being given as integration with respect to μ_E, i.e.,

$$\langle D\mathbb{1}_E, \varphi \rangle = \int_{\mathbb{R}^n} \mathbb{1}_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu_E, \quad \forall \varphi \in C_c^1(\mathbb{R}^n).$$

Therefore we can also say that E is a set of finite perimeter provided that the distributional derivative $D\mathbb{1}_E$ of $\mathbb{1}_E$ is a $\mathbb{R}^n$-valued measure on $\mathbb{R}^n$ with finite total variation on $\mathbb{R}^n$, i.e. $|D\mathbb{1}_E|(\mathbb{R}^n) < \infty$. For this reason we speak of *distributional perimeter* and *distributional Gauss-Green formula* when referring to (1.3).

Remark (Modifications by sets of measure zero). Notice that if E is a set of finite perimeter in $\mathbb{R}^n$ and F in $\mathbb{R}^n$ is such that $|E \Delta F| = 0$, then $\mathbb{1}_E$ and $\mathbb{1}_F$ coincide almost everywhere and hence F is of finite perimeter with $\mu_E = \mu_F$. The converse is also true [Ma1, Remark 12.4] and in particular this means that $P(E)$ is invariant under modifications of E by sets of measure zero, although as we already hinted at the beginning of this section, the size of its topological border ∂E may change drastically.

The notion of relative perimeter arises when we try to find out how much perimeter E holds under certain restriction condition. Let E be a set of C^1 boundary. Then $\nu_E(x)$ is defined for every $x \in \partial E$ and $\mathrm{d}\mu = \nu_E \, \mathrm{d}\mathcal{H}^{n-1}\big|_{\partial E}$ is a locally finite $\mathbb{R}^n$-valued Borel measure. If $\mathcal{H}^{n-1}(\partial E) < \infty$ then μ is a Radon measure satisfying (1.3) and therefore E is of finite perimeter with

$$P(E) = \mathcal{H}^{n-1}(\partial E), \quad \mathrm{d}\mu_E = \nu_E \, \mathrm{d}\mathcal{H}^{n-1}\Big|_{\partial E}.$$

This notion of perimeter agrees with the classical one on sets with C^1 boundary, and if C is a Borel set, we find

$$|\mu_E|(C) = \mathcal{H}^{n-1}(C \cap \partial E),$$

that is, $|\mu_E|(C)$ amounts for the perimeter of the parts of ∂E that lie inside C. Notice that if C is open, then the parts of the boundary of E that coincide with the boundary of C do not contribute to $|\mu_E|(C)$. This expression does not hold in general for any set E of finite perimeter, but the same idea is captured by the notion of *relative perimeter*.

Definition 1.2 (Relative perimeter)**.** Given E a set of finite perimeter in $\mathbb{R}^n$ with Gauss-Green measure μ_E and a measurable set $F \subset \mathbb{R}^n$, we define the relative perimeter of E in F by

$$P(E|F) = |\mu_E|(F).$$

1.1.2 Regularization and set operations

Consider a non-negative function $\rho \in C_c^\infty(B)$ with $\int_{\mathbb{R}^n} \rho = 1$, and define

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right).$$

We now look at the regularization of $\mathbb{1}_E$ via ρ_ε,

$$\mathbb{1}_{E,\varepsilon}(x) = (\rho_\varepsilon * \mathbb{1}_E)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\mathbb{1}_E(y)\,\mathrm{d}y = \int_{E\cap B(x,\varepsilon)} \rho(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Then this function satisfies

$$\mathbb{1}_{E,\varepsilon}(x) = \begin{cases} 1, & |B(x,\varepsilon)\setminus E| = 0, \\ 0, & |B(x,\varepsilon)\cap E| = 0. \end{cases}$$

This means that, intuitively, we expect $\nabla \mathbb{1}_{E,\varepsilon}$ to behave similarly to $-\nu_E$ at the boundary points of E whenever this is an open set with smooth or polyhedral boundary,

$$\nabla \mathbb{1}_{E,\varepsilon}(x + t\nu_E(x)) \approx -\frac{\nu_E(x)}{\varepsilon}, \quad x \in \partial E,\ -\varepsilon < t < \varepsilon.$$

Therefore, it should hold that

$$\int_{\mathbb{R}^n} |\nabla \mathbb{1}_{E,\varepsilon}| \approx \frac{|y \in \mathbb{R}^n :\ \operatorname{dist}(y,\partial E) < \varepsilon|}{\varepsilon} \approx \frac{\varepsilon \mathcal{H}^{n-1}(\partial E)}{\varepsilon} = P(E).$$

We find that this is indeed the case for any set of finite perimeter [Ma1, Proposition 12.20].

Proposition 1.3. *If E is a set of finite perimeter in $\mathbb{R}^n$, then*

$$-\nabla \mathbb{1}_{E,\varepsilon}\,\mathrm{d}x \rightharpoonup^* \mathrm{d}\mu_E, \quad |\nabla \mathbb{1}_{E,\varepsilon}|\,\mathrm{d}x \rightharpoonup^* \mathrm{d}|\mu_E|\,,$$

and in particular,

$$\limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} |\nabla \mathbb{1}_{E,\varepsilon}| = P(E).$$

By regularizing the measure μ_E we obtain $(\mu_E * \rho_\varepsilon)(x) = -\nabla \mathbb{1}_{E,\varepsilon}(x)$, from which the result follows immediately by the weak-* convergence of regularized Radon measures [Ma1, Theorem 4.36]. In particular, this results proves very useful when dealing with unions and intersections of sets of finite perimeter.

Lemma 1.4. *If E and F are sets of finite perimeter in $\mathbb{R}^n$, then $E\cup F$ and $E\cap F$ are sets of finite perimeter in $\mathbb{R}^n$ and for any open set $A \subset \mathbb{R}^n$,*

$$P(E\cup F|A) + P(E\cap F|A) \le P(E|A) + P(F|A).$$

Thanks to Lemma 1.4 we can characterize sets of finite perimeter in $\mathbb{R}$, and in particular we will use this to show the Steiner inequality and the stability of the isoperimetric inequality in dimension $n = 1$.

Proposition 1.5. *If $E \subset \mathbb{R}^n$ is a set of finite perimeter then it is equivalent to a finite union of open intervals lying at mutually positive distance.*

Proof. Let E be a set of finite perimeter in $\mathbb{R}$. We will first assume E to be bounded from below and we will deal with the general case afterwards.

Step 1. Assume $E \subset (a,\infty)$ for some $a \in \mathbb{R}$. We define a function

$$u(x) = -\mu_E((-\infty, x)), \quad x \in \mathbb{R}.$$

Since μ_E can be decomposed into $\mu_E = \mu_E^+ - \mu_E^-$ and $|\mu_E|(-\infty, a) = 0$, both functions $\mu_E^+((-\infty, x))$ and $\mu_E^-((-\infty, x))$ are nondecreasing and therefore Borel for $x \in \mathbb{R}$. Hence u is also a Borel function. Using Fubini's theorem, for any $\varphi \in C_c^1(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} u(x)\varphi'(x)\,\mathrm{d}x &= -\int_{\mathbb{R}} \varphi'(x) \int_{-\infty}^{x} \mathrm{d}\mu_E(y)\,\mathrm{d}x = -\int_{\mathbb{R}}\int_{y}^{\infty} \varphi'(x)\,\mathrm{d}x\,\mathrm{d}\mu_E(y) \\ &= \int_{\mathbb{R}} \varphi(y)\,\mathrm{d}\mu_E(y) = \int_E \varphi'(y)\,\mathrm{d}y = \int_{\mathbb{R}} \varphi'(y)\mathbb{1}_E\,\mathrm{d}y.\end{aligned}$$

In particular, $\int_{\mathbb{R}}(u - \mathbb{1}_E)\varphi' = 0$ for every $\varphi \in C_c^1(\mathbb{R})$, i.e. the weak gradient of $u - \mathbb{1}_E$ vanishes at every φ, which implies (by [Ma1, Lemma 7.5]) that there exists a constant $c \in \mathbb{R}$ such that $u - \mathbb{1}_E = c$ a.e. on $\mathbb{R}$. For $x < a$ the left-hand side is 0, and therefore $c = 0$. Then $u(x) \in \{0, 1\}$ for a.e. $x \in \mathbb{R}$. Since u is left-continuous on $\mathbb{R}$, this holds for all $x \in \mathbb{R}$ and therefore

$$u(x) = \mathbb{1}_F(x), \quad \forall x \in \mathbb{R},$$

for some set $F \subset \mathbb{R}$ with $|E\Delta F| = 0$. Hence

$$\limsup_{h\to 0} |\mathbb{1}_F(x+h) - \mathbb{1}_F(x)| \le |\mu_E|(\{x\}), \quad \forall x \in \mathbb{R}.$$

Tis means that $|\mu_E|(\{x\}) \ge 1$ at every discontinuity point of $\mathbb{1}_F$. Now, since E is of finite perimeter and $|\mu_E|$ is finite, the set of discontinuity points of $\mathbb{1}_F$ is finite, and thus F is a finite union of intervals lying at mutually positive distance.

Step 2. Let E be a set of finite perimeter in $\mathbb{R}$, possibly unbounded. Define $E_k = E \cap (-k, \infty)$, for $k \in \mathbb{N}$. Then E_k is bounded from below and it has finite perimeter by Lemma 1.4. By applying the above argument to these sets we conclude the proof. □

1.1.3 The constrained isoperimetric problem

The relative perimeter enjoys a couple of interesting properties that allow for an initial approach towards the isoperimetric problem. The first one is its lower semicontinuity in the L^1-convergence of measurable sets, with which $E_h \longrightarrow E$ whenever

$$|E\Delta E_h| \longrightarrow 0, \quad \text{as } h \to \infty.$$

As a matter of fact, it is only necessary for our purpose to consider L^1-local convergence, according to which $E_h \longrightarrow^{\mathrm{loc}} E$ if

$$|K \cap (E\Delta E_h)| \longrightarrow 0, \quad \text{as } h \to \infty$$

for every compact set $K \subset \mathbb{R}^n$. We have the following result.

Theorem 1.6 (Lower semicontinuity of relative perimeter)**.** *Let $\{E_h\}_{h=1}^{\infty}$ be a sequence of sets of finite perimeter in $\mathbb{R}^n$ with $E_h \longrightarrow^{\mathrm{loc}} E$. Then E is of finite perimeter in $\mathbb{R}^n$ and for any open set $A \subset \mathbb{R}^n$ we have*

$$P(E|A) \le \liminf_{h\to\infty} P(E_h|A).$$

Another impressive feature of the perimeter, which we aim to apply later in combination with Theorem 1.6, is the following compactness result. The relevance of these results lies in that together, they complete the requirements needed to apply the Direct Method to several minimization problems regarding the perimeter and relative perimeter.

Theorem 1.7 (Compactness of perimeter)**.** *If $R > 0$ and $\{E_h\}_{h=1}^{\infty}$ is a sequence of sets of finite perimeter in $\mathbb{R}^n$ such that*

$$\sup_{h\in\mathbb{N}} P(E_h) < \infty \quad \textit{and} \quad E_h \subset B_R \quad \forall h \in \mathbb{N},$$

then there exists a set of finite perimeter E in $\mathbb{R}^n$ and $h(k) \to \infty$ as $k \to \infty$ with

$$E_{h(k)} \longrightarrow E, \quad E \subset B_R.$$

We now introduce the *constrained isoperimetric problem*, a simple version of the isoperimetric problem where we consider our competitor sets to be bounded by some ball. This problem is very closely related to the isoperimetric inequality, and in fact we will use it in proving the latter.

Proposition 1.8. *If $R > 0$ and m is such that $0 < m < |B|\, R^n$, then there exists a set of finite perimeter $E \subset B_R$ that is a minimizer for the variational problem*

$$\inf \{P(E) : \ E \subset B_R, \ |E| = m\}. \tag{1.4}$$

Proof. Notice that the condition on m makes the class of competitors nonempty, and the perimeter is always nonnegative (thus the problem has a lower bound). Let α denote the infimum in (1.4) and let $\{E_h\}_{h=1}^{\infty}$ be a minimizing sequence with $P(E_h) \to \alpha$ as $h \to \infty$. By Theorem 1.7, there exists a set $E \subset B_R$ of finite perimeter with $E_h \to E$ up to a subsequence. Moreover, $|E| = \lim_{h\to\infty} |E_h| = m$ and by Theorem 1.6,

$$\alpha \leq P(E) \leq \liminf_{h\to\infty} P(E_h) = \alpha,$$

completing the proof. □

Remark. Dropping the hypothesis $E \subset B_R$ leads us to the existence of optimal sets in the Euclidean isoperimetric problem and this task is far from trivial. In particular we cannot approach that problem using the Direct Method since the compactness result can no longer be applied.

In a much similar vein to how we proved Proposition 1.4 we can show the existence of minimizers for the *relative isoperimetric problem*. The only difference is that we will now have to apply Lemma 1.4 in order to bound uniformly the perimeters of the sets in the minimizing sequence so that we can apply Theorem 1.7.

Proposition 1.9. *If A is an open, bounded set of finite perimeter and $m \in (0, |A|]$, then there exists a set of finite perimeter $E \subset A$ such that E is a minimizer for the problem*

$$\inf \{P(E|A) : \ E \subset A, \ |E| = m\}.$$

1.1.4 Approximation by smooth or polyhedral sets

Let us begin with an example as a motivation for this section. This can be found in [Ma2, p. 373]

Example 1.10. Let $\{\varepsilon_h\}_{h=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{h=1}^{\infty} \varepsilon_h^{n-1} < \infty$, and let $\{x_h\}_{h=1}^{\infty}$ list all the elements in $\mathbb{Q}^n \cap [0,1]^n$. Define the open set

$$E = \bigcup_{h \in \mathbb{N}} B(x_h, \varepsilon_h).$$

Then the closure of E contains $[0,1]^n$ and as long as $|E| < 1$ (which we can obtain if ε_h is small enough) we have $|[0,1]^n \setminus E| > 0$, and therefore $\mathcal{H}^{n-1}(\partial E) = \infty$. Despite this, E is a set of finite perimeter. We can see this by setting $E_N = \bigcup_{h=1}^{N} B(x_h, \varepsilon_h)$, so that $E_N \to E$, and then by the lower semicontinuity and Lemma 1.4,

$$P(E) \leq \liminf_{N\to\infty} P(E_N) \leq \liminf_{N\to\infty} P(B(x_h, \varepsilon_h)) = \liminf_{N\to\infty} n\,|B| \sum_{h=1}^{N} \varepsilon_h^{n-1} < \infty.$$

This example illustrates how sets of finite perimeter can be highly irregular, and in order to handle their complexity the following approximation result is crucial.

Theorem 1.11 (Approximation by smooth or polyhedral sets)**.** *If E is a set of finite perimeter in $\mathbb{R}^n$, then there exists a sequence $\{E_h\}_{h=1}^{\infty}$ of bounded open sets with smooth or polyhedral boundary so that $E_h \to E$ and $P(E_h) \to P(E)$.*

The theorem follows from the following instance of the Coarea Formula, which we will also need in future proofs.

Theorem 1.12 (Coarea Formula for the total variation)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be smooth or piecewise affine, and let $\{f > t\}$ denote the set of points x such that $f(x) > t$, for $t \in \mathbb{R}$. Then*

$$\int_{\mathbb{R}^n} |\nabla f| = \int_{\mathbb{R}} P(\{f > t\})\,\mathrm{d}t.$$

Additionally, in proving stability for the isoperimetric inequality we will need to use extensively the following particular example of the Coarea Formula for rectifiable sets, which describes the behaviour of the reduced boundary (see Definition 1.15 below) of a set of finite perimeter upon being sliced.

Theorem 1.13. *Let E be a set of finite perimeter in $\mathbb{R}^n$, $n \geq 2$. Then, for a.e. $t \in \mathbb{R}$, the section E_t is a set of finite perimeter in $\mathbb{R}^{n-1}$ and its reduced boundary $\mathcal{F}E_t$ is $\mathcal{H}^{n-2}$-equivalent to the t-section $(\mathcal{F}E)_t$ of the reduced boundary of E. Furthermore, for every Borel function $g : \mathbb{R}^n \longrightarrow [0,\infty]$ we have*

$$\int_{\mathcal{F}E} g\sqrt{1 - |\nu_E \cdot e_1|^2}\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{R}} \mathrm{d}t \int_{(\mathcal{F}E)_t} g(t,z)\,\mathrm{d}\mathcal{H}^{n-2}(z), \tag{1.5}$$

where $\nu_E \cdot e_1$ denotes the first component of the outer unit normal vector to E.

1.1.5 Density properties and the reduced boundary

Every measurable set in $\mathbb{R}^n$ enjoys certain basic density properties. We can summarize these by saying that a.e. point of E has density one, while a.e. point of $\mathbb{R}^n \setminus E$ has density zero, with respect to E. When E has a smooth boundary, then we could expect that the points in ∂E have density $1/2$. Indeed,

$$\frac{|E \cap B(x,r)|}{|B(x,r)|} = \left| B \cap \frac{E-x}{r} \right|,$$

and if $x \in \partial E$, and $\nu_E(x)$ denotes the outer unit normal to E, then

$$\frac{E-x}{r} \longrightarrow^{\text{loc}} \{y \in \mathbb{R}^n : \ \nu_E(x) \cdot (y-x) \leq 0\}.$$

A crucial property of sets of finite perimeter is that this density property of boundary points remains valid in a sense, provided that the classical notions of *boundary point* and *outer normal* are suitably extended.

For this last notion we can use the Polar Decomposition Theorem for Radon measures on the Gauss-Green measure μ_E of a given set $E \subset \mathbb{R}^n$ of finite perimeter to ensure the existence of a Borel function $\nu_E : \mathbb{R}^n \longrightarrow \mathbb{S}^{n-1}$ such that $|\nu_E(x)| = 1$ for $|\mu_E|$-a.e. $x \in \mathbb{R}^n$, and $\mu_E = |\nu_E| \, \mathrm{d}\mu_E$. The Besicovitch Differentiation Theorem for Radon measures then yields (1.6) and we have the following definition.

Definition 1.14. Given E a set of finite perimeter in $\mathbb{R}^n$ with Gauss-Green measure μ_E, we call ν_E the *measure-theoretic outer normal* to E, satisfying

$$\nu_E(x) = \lim_{r \to 0} \frac{\mu_E(B(x,r))}{|\mu_E|\,(B(x,r))}, \tag{1.6}$$

for $|\mu_E|$-a.e. $x \in \mathbb{R}^n$.

We use this notation since ν_E should be regarded as the outer normal to the set E. In fact, when E has smooth or polyhedral boundary, ν_E coincides with the classical notion of outer normal to E. With this tool in hand we finally have a way to state which points of ∂E essentially contribute to $P(E)$.

Definition 1.15. Given a set E of finite perimeter in $\mathbb{R}^n$, we define its *reduced boundary* $\mathcal{F}E$ as the set of points $x \in \mathbb{R}^n$ where (1.6) holds true, $|\nu_E(x)| = 1$ and $|\mu_E|\,(B(x,r)) > 0$.

When E is open and has C^1 boundary, it has $\mathcal{F}E = \partial E$, and if E is of finite perimeter, it turns out that concerning density properties, it behaves similarly to a set of smooth boundary on its reduced boundary. For $\lambda \in [0,1]$, let us denote by $E^{(\lambda)}$ the set of points $x \in \mathbb{R}^n$ having density λ with respect to E, i.e., $x \in E^{(\lambda)}$ if

$$\lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = \lambda.$$

We use the notation $\partial_{1/2} E$ for $E^{(1/2)}$, and introduce the *essential boundary* $\partial^{\mathrm{e}} E$ of E by setting

$$\partial^{\mathrm{e}} E = \mathbb{R}^n \setminus [E^{(0)} \cup E^{(1)}].$$

The following result by Federer [Ma1, Theorem 16.2] relates $\mathcal{F}E$ to the sets $\partial_{1/2} E$ and $\partial^{\mathrm{e}} E$.

Theorem 1.16 (Federer's theorem)**.** *If E is a set of finite perimeter in $\mathbb{R}^n$, then $\mathcal{F}E \subset \partial_{1/2}E \subset \partial^e E$, with*

$$\mathcal{H}^{n-1}\left(\partial^e E \setminus \mathcal{F}E\right) = 0.$$

In particular, the theorem states that the three sets $\mathcal{F}E, \partial_{1/2}E$ and $\partial^{\mathrm{e}}E$ are $\mathcal{H}^{n-1}$-equivalent. We will use this result extensively when we address the problem of stability in the anisotropic isoperimetric inequality. The following theorem on the structure of the reduced boundary and the Gauss-Green measure of E is one of the cornerstones in the theory of sets of finite perimeter.

Theorem 1.17 (De Giorgi's structure theorem)**.** *If E is a set of finite perimeter in $\mathbb{R}^n$, then the Gauss-Green measure μ_E of E satisfies*

$$\mathrm{d}\mu_E = \nu_E \,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}E}, \quad \mathrm{d}|\mu_E| = \mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}E}, \tag{1.7}$$

and the generalized following Gauss-Green formula holds,

$$\int_E \nabla\varphi(x)\,\mathrm{d}x = \int_{\mathcal{F}E} \varphi(x)\nu_E(x)\,\mathrm{d}\mathcal{H}^{n-1}(x), \quad \forall\varphi \in C_c^1(\mathbb{R}^n). \tag{1.8}$$

Moreover, there exist countably many C^1-hypersurfaces M_h in $\mathbb{R}^n$, compact sets $K_h \subset M_h$ and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$ such that

$$\mathcal{F}E = F \cup \bigcup_{h\in\mathbb{N}} K_h,$$

and for every $x \in K_h$ and every vector v tangent to M_h, $\nu_E(x)\cdot v = 0$.

In modern terms, the second part of Theorem 1.17 can be stated by saying that $\mathcal{F}E$ is an $(n-1)$-rectifiable set. We also have the following theorem regarding set operations and Gauss-Green measures.

Theorem 1.18. *If E and F are sets of finite perimeter and*

$$\{\nu_E = \nu_F\} = \{x \in \mathcal{F}E \cap \mathcal{F}F :\ \nu_E(x) = \nu_F(x)\},$$
$$\{\nu_E = -\nu_F\} = \{x \in \mathcal{F}E \cap \mathcal{F}F :\ \nu_E(x) = -\nu_F(x)\},$$

then $E\cap F$, $E\setminus F$ and $E\cup F$ are sets of finite perimeter, with

$$\mu_{E\cap F} = \mu_E\big|_{F^{(1)}} + \mu_F\big|_{E^{(1)}} + \nu_E\mathcal{H}^{n-1}\Big|_{\{\nu_E=\nu_F\}}, \tag{1.9}$$

$$\mu_{E\setminus F} = \mu_E\big|_{F^{(0)}} - \mu_F\big|_{E^{(1)}} + \nu_E\mathcal{H}^{n-1}\Big|_{\{\nu_E=-\nu_F\}}, \tag{1.10}$$

$$\mu_{E\cup F} = \mu_E\big|_{F^{(0)}} + \mu_F\big|_{E^{(0)}} + \nu_E\mathcal{H}^{n-1}\Big|_{\{\nu_E=\nu_F\}}, \tag{1.11}$$

or equivalently,

$$\nu_{E\cap F}\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}(E\cap F)} = \nu_E\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}E\cap F^{(1)}} + \nu_F\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}F\cap E^{(1)}} + \nu_E\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\{\nu_E=\nu_F\}}, \tag{1.12}$$

$$\nu_{E\setminus F}\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}(E\setminus F)} = \nu_E\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{(\mathcal{F}E)\setminus F} - \nu_F\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}F\cap E^{(1)}} + \nu_E\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\{\nu_E=-\nu_F\}}, \tag{1.13}$$

$$\nu_{E\cup F}\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}(E\cup F)} = \nu_E\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{(\mathcal{F}E)\setminus F} + \nu_F\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{(\mathcal{F}F)\setminus E} + \nu_E\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\{\nu_E=\nu_F\}}. \tag{1.14}$$

Moreover, the reduced boundaries satisfy

$$\mathcal{F}(E\cap F) = \left(F^{(1)}\cap\mathcal{F}E\right)\cup\left(E^{(1)}\cap\mathcal{F}F\right)\cup\{\nu_E=\nu_F\}, \tag{1.15}$$

$$\mathcal{F}(E\setminus F) = ((\mathcal{F}E)\setminus F)\cup\left(E^{(1)}\cap\mathcal{F}F\right)\cup\{\nu_E=-\nu_F\}, \tag{1.16}$$

$$\mathcal{F}(E\cup F) = ((\mathcal{F}E)\setminus F)\cup((\mathcal{F}F)\setminus E)\cup\{\nu_E=\nu_F\}. \tag{1.17}$$

up to $\mathcal{H}^{n-1}$-null sets.

The proofs for these results can be found in [Ma1, pp. 167-187]. In the particular case that $F\subset E$, we have $F^{(1)}\subset E^{(1)}$ and therefore (1.12) and (1.15) reduce to

$$\mathcal{F}F = [\mathcal{F}F\cap\mathcal{F}E]\cup[\mathcal{F}F\cap E^{(1)}], \tag{1.18}$$

$$\nu_F(x) = \nu_E(x), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in\mathcal{F}F\cap\mathcal{F}E, \tag{1.19}$$

again up to $\mathcal{H}^{n-1}$-null sets. Additionally, when $|E\cap F|=0$, since $\partial_{1/2}E\cap\partial_{1/2}F\subset(E\cup F)^{(1)}$, we have, by Theorem 1.16,

$$\mathcal{H}^{n-1}\left(\mathcal{F}(E\cup F)\cap(\mathcal{F}E\cap\mathcal{F}F)\right)=0,$$

and (1.14) turns into

$$\nu_{E\cup F}\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{\mathcal{F}(E\cup F)} = \nu_E\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{(\mathcal{F}E)\setminus F} + \nu_F\,\mathrm{d}\mathcal{H}^{n-1}\Big|_{(\mathcal{F}F)\setminus E} \tag{1.20}$$

This implies as well (by (1.14)) that $\nu_E=-\nu_F$ at $\mathcal{H}^{n-1}$-a.e. point of $\mathcal{F}E\cap\mathcal{F}F$ when $|E\cap F|=0$.

2 Euclidean isoperimetric inequality

Having in hand all the necessary tools to handle sets of finite perimeter we address the isoperimetric inequality.

Theorem 2.1 (Isoperimetric inequality)**.** *If E is a measurable subset of $\mathbb{R}^n$, $n \geq 1$, and $0 < |E| < \infty$, then*

$$n\,|B|^{1/n}\,|E|^{(n-1)/n} \leq P(E). \tag{2.1}$$

Equality holds if and only if E coincides with a ball up to a set of measure zero.

The classical proof of Theorem 2.1 is based on the powerful tool of Steiner symmetrization. We will discuss it in section 2.1, and the proof of the isoperimetric inequality will follow in section 2.2.

Notice that we have already proved existence of minimizers in the constrained setting in Proposition 1.8. In order to extend this to the general case and prove Theorem 2.1, we will first characterize the minimizers of the constrained problem (1.4), and then argue by contradiction that no minimizer for the unbounded case is unbounded.

2.1 Steiner symmetrization

The characterization of the equality cases in the isoperimetric inequality relies on Steiner symmetrization. The idea behind this technique is to symmetrize a set E with respect to a certain hyperplane while preserving its volume. In this way we obtain another set E^s with smaller perimeter but the same volume as E, and thus E^s acts as a better competitor than E in the isoperimetric problem. If symmetrizing with respect to any hyperplane can no longer decrease the perimeter, then the resulting set must be symmetric in every direction and equivalent to a convex set, and therefore it must be equivalent to a ball.

Given a measurable set E, we fix a direction $\nu \in \mathbb{S}^{n-1}$ and construct the *Steiner symmetral E^s of E with respect to ν* as follows. Let $H = \{y \in \mathbb{R}^n : \ y \cdot \nu = 0\}$ be the hyperplane perpendicular to ν, and define

$$\begin{aligned} m : H &\longrightarrow [0,\infty] \\ y &\longmapsto \mathcal{H}^1\left(E \cap \{y + t\nu : \ t \in \mathbb{R}\}\right). \end{aligned} \tag{2.2}$$

By Fubini's Theorem m is measurable, and $m \in L^1\left(\mathcal{H}^{n-1}\big|_H\right)$ as long as $|E| < \infty$ since $\int m \, \mathrm{d}\mathcal{H}^{n-1}\big|_H = |E|$. We set

$$E^s = \left\{ y + t\nu : \ y \in H, \ m(y) > 0, \ -\frac{m(y)}{2} \leq t \leq \frac{m(y)}{2} \right\}.$$

This corresponds to centering, at every $y \in H$, an interval of length $m(y)$, the length of the section of E that intersects a line in the ν direction and passes through y. By construction, E^s is measurable and symmetric with respect to the hyperplane H. Additionally, $|E^s| = |E|$.

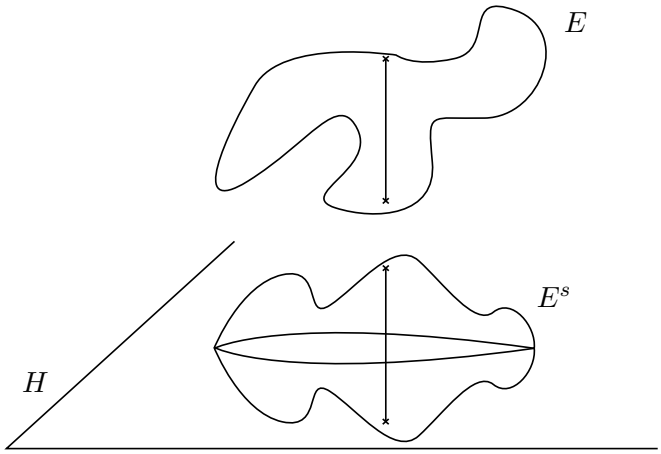


FIGURE 1: Steiner symmetrization.

Theorem 2.2 (Steiner inequality)**.** *For every set E of finite measure we have*

$$P(E^s) \leq P(E). \tag{2.3}$$

Moreover,

(i) if equality holds, then for $\mathcal{H}^{n-1}$-a.e. $y \in H$ the section $E \cap \{y + t\nu : t \in \mathbb{R}\}$ is $\mathcal{H}^1$-equivalent to an interval;

(ii) if E is convex, then equality holds if and only if there exists $c \in \mathbb{R}$ such that $E = E^s + c\nu$.

Proof. We will first show the inequality in dimension $n = 1$, using Proposition 1.5. Then we will prove it for bounded sets with polyhedral boundary and use Theorem 1.11 to extend this to an arbitrary set of finite perimeter. Finally, we will prove *(i)* and *(ii)*.

Step 1. When $n = 1$, for any set E of finite measure, $E^s = (-|E|/2, |E|/2)$ and the Steiner inequality is the isoperimetric inequality. By Proposition 1.5, if $P(E) < \infty$ and $|E| < \infty$, then

$$E = \bigcup_{i=1}^{N} (a_i, b_i), \quad b_i < a_{i+1}, \quad N \geq 1,$$

up to a set of measure zero. Therefore $\mathcal{F}E = \{a_i, b_i : i = 1, ..., N\}$, and $P(E) = \mathcal{H}^0(\mathcal{F}E) = 2N \geq 2$, with equality holding if and only if $N = 1$, i.e., when E is an interval. Notice that this proves Theorem 2.1 in dimension 1.

Step 2. Let $n \geq 2$ and define $m : H \longrightarrow [0, 1]$ as in (2.2). Let us examine the case in which E is bounded and has a polyhedral boundary, whose outer normal field is never orthogonal to the direction of symmetrization ν. By the Implicit Function Theorem, we

can find a partition of $\{m > 0\} \subset H$ into finitely many $(n-1)$-dimensional polyhedral sets $C_1, ..., C_M$,

$$\{y \in H : \ m(y) > 0\} = \bigcup_{j=1}^{M} C_j, \quad M \in \mathbb{N},$$

and for each $j \in 1, ..., M$, there exist finitely many affine functions $g_{j,k}, f_{j,k} : C_j \longrightarrow \mathbb{R}$, $k = 1, ..., N(j)$, satisfying

$$E \cap (C_j \times \mathbb{R}) = \left\{ y + t\nu : \ y \in C_j, \ t \in \bigcup_{k=1}^{N(j)} (g_{j,k}(y), f_{j,k}(y) \right\},$$
$$2N(j) = \mathcal{H}^0 \left(\mathcal{F}E \cap \{y + t\nu : \ t \in \mathbb{R}\} \right), \quad \forall y \in C_j.$$

By construction, if $y \in C_j$, then

$$m(y) = \sum_{k=1}^{N(j)} f_{j,k}(y) - g_{j,k}(y). \tag{2.4}$$

Now we prove the inequality by a convexity argument. Since E_h fits in between the graphs of $-m/2$ and $m/2$, we have

$$P(E^s) = 2 \int_{\{m>0\}} \sqrt{1 + \left| \frac{\nabla m}{2} \right|^2} = \sum_{j=1}^{M} \int_{C_j} \sqrt{4 + |\nabla m|^2} \, \mathrm{d}\mathcal{H}^{n-1}.$$

On the other hand,

$$\begin{aligned} P(E) &= \sum_{j=1}^{M} \mathcal{H}^{n-1} \left(\mathcal{F}E \cap [C_j \times \mathbb{R}] \right) \\ &= \sum_{j=1}^{M} \int_{C_j} \sum_{k=1}^{N(j)} \sqrt{1 + |\nabla f_{j,k}|^2} + \sqrt{1 + |\nabla g_{j,k}|^2} \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

Now we recall (2.4) and by convexity, for any point of C_j,

$$\begin{aligned} \sum_{k=1}^{N(j)} \sqrt{1 + |\nabla f_{j,k}|^2} + \sqrt{1 + |\nabla g_{j,k}|^2} &\geq 2 \sum_{k=1}^{N(j)} \sqrt{1 + \left| \frac{\nabla f_{j,k} - \nabla g_{j,k}}{2} \right|^2} \\ &= 2N(j) \left(\frac{1}{N(j)} \sum_{k=1}^{N} \sqrt{1 + \left| \frac{\nabla f_{j,k} - \nabla g_{j,k}}{2} \right|^2} \right) \\ &\geq 2N(j) \sqrt{1 + \left| \frac{1}{N(j)} \sum_{k=1}^{N} \frac{\nabla f_{j,k} - \nabla g_{j,k}}{2}^2 \right|} \\ &= \sqrt{4N(j)^2 + |\nabla m|^2} \geq \sqrt{4 + |\nabla m|^2}. \end{aligned}$$

Therefore $P(E) \geq P(E^s)$. In fact, we can even do better with this argument, we can control the $\mathcal{H}^{n-1}$-size of the set D of those points $y \in H$ for which $\{y + t\nu : \ t \in \mathbb{R}\} \cap E$ is not a segment by how much the perimeter of E gets reduced after being symmetrized. Let D be such set, namely the union of those C_j's such that $N(j) \geq 2$. We have

$$P(E) - P(E^s) \geq \sum_{j=1}^{M} \int_{C_j} \sqrt{4N(j)^2 + |\nabla m|^2} - \sqrt{4 + |\nabla m|^2} \, \mathrm{d}\mathcal{H}^{n-1}$$

$$= \sum_{j=1}^{M} \int_{C_j} \frac{4\left(N(j)^2-1\right)}{\sqrt{4N(j)^2+|\nabla m|^2}+\sqrt{4+|\nabla m|^2}}\,\mathrm{d}\mathcal{H}^{n-1}$$
$$\geq 2\sum_{j=1}^{M} \int_{C_j\cap D} \frac{1}{\sqrt{4N(j)^2+|\nabla m|^2}}\,\mathrm{d}\mathcal{H}^{n-1}.$$

Since we already know that

$$\sum_{j=1}^{M} \int_{C_j} \sqrt{4N(j)^2+|\nabla m|^2} \leq P(E),$$

we can deduce, using the Hölder inequality, that

$$2\mathcal{H}^{n-1}(D)^2 \leq \left(P(E)-P(E^s)\right)P(E).$$

Step 3. Now let E be a set of finite perimeter. By Theorem 1.11 we find a sequence of bounded open sets E_h with polyhedral boundary and such that

$$|E_h \Delta E| \to 0, \quad P(E_H) \to P(E). \tag{2.5}$$

As ν_{E_h} takes only finitely many values, we can assume that $\nu_E \cdot \nu \neq 0$ on $\mathcal{F}E_h$ up to rotating E_h by a rotation sufficiently close to the identity. In this way E_h satisfies the assumptions of *Step 2*, and therefore

$$P(E_h) \geq P(E_h^s), \tag{2.6}$$
$$2\mathcal{H}^{n-1}(D_h)^2 \leq \left(P(E_h)-P(E_h^s)\right)P(E_h), \tag{2.7}$$

where $D_h \subset H$ is the set of those $y \in H$ such that the section $E_h(y) = E_h \cap \{y+t\nu : t \in \mathbb{R}\}$ is not a segment. Denote by $E(y)$ the corresponding section of E. Then

$$|E_h \Delta E| = \int_H \mathcal{H}^1\left(E_h(y)\Delta E(y)\right)\mathrm{d}\mathcal{H}^{n-1}(y)$$
$$\geq \int_H |m_h(y)-m(y)|\,\mathrm{d}\mathcal{H}^{n-1}(y) = \left|E_h^s \Delta E^s\right|.$$

In particular $E_h^s \to E^s$ and by lower semicontinuity, $P(E^s) \leq \liminf_{h\to\infty} P(E_h^s)$. As $P(E_h) \to P(E)$, we deduce that $P(E^s) \leq P(E)$ by (2.6).

Step 4. If we now assume that equality holds in the Steiner inequality for E, by passing to the limit in (2.7) we find that $\mathcal{H}^{n-1}(D_h) \to 0$ as $h \to \infty$. Now recall that $\mathcal{H}^1\left(E_h(y)\Delta E(y)\right) \to 0$ for $\mathcal{H}^{n-1}$-a.e. $y \in \{m(y)>0\}$. By the lower semicontinuity, we find

$$\liminf_{h\to\infty} \mathbb{1}_{\{m_h>0\}\setminus D_h}(y)\mathcal{H}^0\left(\mathcal{F}E_h(y)\right) \geq \mathbb{1}_{\{m>0\}}(y)\mathcal{H}^0\left(\mathcal{F}E(y)\right),$$

for $\mathcal{H}^{n-1}$-a.e. $y \in H$. By Fatou's Lemma,

$$\int_{\{m>0\}} \mathcal{H}^0\left(\mathcal{F}E(y)\right)\mathrm{d}\mathcal{H}^{n-1}(y) \leq \liminf_{h\to\infty} \int_{\{m_h>0\}\setminus D_h} \mathcal{H}^0\left(\mathcal{F}E_h(y)\right)\mathrm{d}\mathcal{H}^{n-1}(y)$$
$$= 2\mathcal{H}^{n-1}\left(\{m>0\}\right).$$

By *Step 1* we have $\mathcal{H}^0\left(\mathcal{F}E(y)\right) \geq 2$. Thus it turns out that $\mathcal{H}^0\left(\mathcal{F}E(y)\right) = 2$ for $\mathcal{H}^{n-1}$-a.e. $y \in \{m>0\}$, and every section $E(y)$ is $\mathcal{H}^1$-equivalent to a segment, which

proves *(i)*. Additionally, these arguments can be adapted to show a relative version of the Steiner inequality,

$$P(E|A\tilde{\times}\mathbb{R}) \geq P(E^s|A\tilde{\times}\mathbb{R}), \quad \forall A \subset H \text{ open}, \tag{2.8}$$

where $A\tilde{\times}\mathbb{R}$ denotes the set $\{y + t\nu : \ y \in A, \ t \in \mathbb{R}\}$.

Step 5. To conclude the proof of the theorem, let E be convex and such that $P(E) = P(E^s)$. We can find an open convex set $D \subset H$ and a pair a concave nonnegative functions $\psi_1, \psi_2 : D \longrightarrow [0, \infty)$ with

$$E = \{y + t\nu : \ y \in H, \ -\psi_1(y) < t < \psi_2(y)\}.$$
$$E^s = \left\{y + t\nu : \ y \in H, \ -\frac{\psi_1(y) + \psi_2(y)}{2} < t < \frac{\psi_1(y) + \psi_2(y)}{2}\right\}.$$

Let D' be an open set compactly contained in D. As $P(E) = P(E^s)$ and (2.8) holds on both $D'\tilde{\times}\mathbb{R}$ and $(F \setminus D')\tilde{\times}\mathbb{R}$, we deduce that

$$P(E|D'\tilde{\times}\mathbb{R}) = P(E^s|D'\tilde{\times}\mathbb{R}).$$

On the other hand, since ψ_1 and ψ_2 are locally Lipschitz, these perimeters can be expressed via the elementary formulas

$$P(E|D'\tilde{\times}\mathbb{R}) = \int_{D'} \sqrt{1 + |\nabla\psi_1|^2} + \sqrt{1 + |\nabla\psi_2|^2}\, \mathrm{d}\mathcal{H}^{n-1},$$
$$P(E^s|D'\tilde{\times}\mathbb{R}) = \int_{D'} \sqrt{1 + \left|\frac{\nabla\psi_1 + \nabla\psi_2}{2}\right|^2}\, \mathrm{d}\mathcal{H}^{n-1}.$$

By strict convexity, there exists $c \in \mathbb{R}$ such that $\psi_1 - \psi_2 = c$ on every D', making $E = c\nu + E^s$. □

2.2 Proof of the Euclidean isoperimetric inequality

Before getting to the proof of Theorem 2.1 we are going to need the following lemma, which can be found in [Ma1, Lemma 14.6].

Lemma 2.3. *If E is a set of finite perimeter in $\mathbb{R}^n$ such that for a.e. $y \in H$ the section $E \cap \{y + t\nu : \ t \in \mathbb{R}\}$ is $\mathcal{H}^1$-equivalent to a segment, then for every $y \in H$, the section $E^{(1)} \cap \{y + t\nu : \ t \in \mathbb{R}\}$ of $E^{(1)}$ is a segment.*

Proof of Theorem 2.1. Let $m, R > 0$ with $m < |B|\, R^n$ and consider a minimizer E to the constrained problem (1.4). Given any direction $\nu \in \mathbb{S}^{n-1}$, E^s is a better competitor for the problem, and since E is a minimizer, $P(E^s) = P(E)$ and equality holds in the Steiner inequality. We can therefore apply *(i)* in Theorem 2.2 to obtain that $\mathcal{H}^{n-1}$-a.e. one-dimensional section of E in the direction ν is $\mathcal{H}^1$-equivalent to a segment for any given $\nu \in \mathbb{S}^{n-1}$. By Lemma 2.3, this fact extends to *every* one-dimensional section of $E^{(1)}$, and since this set is symmetric with respect to any direction, it is convex. Indeed, if $x, y \in E$, it suffices to consider $\nu = \frac{x-y}{|x-y|}$: by the discussion above, the segment joining x and y is contained in a one-dimensional section of E.

Therefore we know that $E^{(1)}$ is convex, equivalent to E and it has $P(E^{(1)}) = P(E)$, so equality holds in the Steiner inequality for $E^{(1)}$. By *(ii)* in Theorem 2.2, for every $\nu \in \mathbb{S}^{n-1}$ we can find a constant $c_\nu \in \mathbb{R}$ such that

$$E^{(1)} = c_\nu \nu + E_\nu^{(1)s},$$

where $E_\nu^{(1)s}$ denotes the Steiner symmetral of $E^{(1)}$ with respect to ν. This means that $E^{(1)}$ and $E_\nu^{(1)s}$ are the same set up to a translation by $c_\nu \nu$. Since this holds for every $\nu \in \mathbb{S}^{n-1}$, $E^{(1)}$ is symmetric by reflection with respect to every direction, and because it is convex we conclude it must be a ball. Hence

$$\inf \{P(E) : \ E \subset B_R, \ |E| = m\} = P(E^{(1)}) = n\,|B|^{1/n}\, m^{(n-1)/n},$$

and the infimum is attained if and only if E is equivalent to a ball, which immediately implies that

$$n\,|B|^{1/n} = \inf \left\{ \frac{P(E)}{|E|^{(n-1)/n}} : \ E \text{ is bounded} \right\},$$

with attainment if and only if E is equivalent to a ball. By approximation with bounded sets (Theorem 1.11),

$$n\,|B|^{1/n} = \inf \left\{ \frac{P(E)}{|E|^{(n-1)/n}} : \ |E| < \infty \right\}.$$

Therefore the isoperimetric inequality holds, and in the bounded case the minimizers are balls.

Finally, we are left to prove that this is also the case if equality holds for some unbounded set E. With these assumptions we can repeat the arguments based on the Steiner inequality to find that $E^{(1)}$ has to be equivalent to a ball, against the assumption that it is unbounded. □

3 Stability of the Euclidean isoperimetric inequality

A natural question that arises from any functional inequality concerns its stability. For the isoperimetric inequality, we may ask ourselves whether a set E for which the equality in (2.1) fails by a small margin must be geometrically close to a ball or not. This is, we want to find a quantitative version of (2.1). For an analytic statement we first need to define a few concepts. From here on we may adapt the notation $n' = n/(n-1)$, so that the isoperimetric inequality takes the form

$$n\,|B|^{1/n}\,|E|^{1/n'} \leq P(E).$$

Definition 3.1. If E is a set of finite volume in $\mathbb{R}^n$, we define its *isoperimetric deficit* by

$$\delta(E) := \frac{P(E)}{n\,|B|^{1/n}\,|E|^{1/n'}} - 1. \tag{3.1}$$

Of course, $\delta(E) \geq 0$ by (2.1), with attainment of the equality whenever E is equivalent to a ball. Additionally, $\delta(E)$ defines a functional $E \mapsto \delta(E)$ that is scale invariant and translation invariant. The purpose of $\delta(E)$ is to measure the failure of E being optimal in the isoperimetric inequality. It is reasonable to think that in some sense it controls the distance between E and a ball, if there is stability, so we aim to relate this to a more direct notion of the distance between E and a ball.

Definition 3.2. Given a set E of finite volume in $\mathbb{R}^n$, we define its *Fraenkel asymmetry* by

$$A(E) := \inf\left\{\frac{|E\Delta B(x,r)|}{|E|} :\ |B(x,r)| = |E|\,,\ x \in \mathbb{R}^n\right\}. \tag{3.2}$$

Furthermore, if $F \subset \mathbb{R}^n$, we define the *relative Fraenkel asymmetry of E with respect to F* by

$$A(E|F) := \inf\left\{\frac{|E\Delta B(x,r)|}{|E|} :\ |B(x,r)| = |E|\,,\ x \in F\right\}. \tag{3.3}$$

The Fraenkel asymmetry is a nonnegative, scale and translation invariant functional, defined on the space of sets of finite volume. Additionally, $0 \leq A(E) \leq 2$, with attainment of the equality $A(E) = 0$ only when E is equivalent to a ball. With both the Fraenkel asymmetry and the isoperimetric deficit at hand, we seek to control $A(E)$ by $\delta(E)$ in some way to establish stability of the isoperimetric inequality.

The first results in this direction were due to Bernstein [Be], Bonnesen [Bo] and Osserman [Os1], and covered the problem showing stability in dimension $n = 2$. In the particular case of convex sets, a result in $\mathbb{R}^n$ was first established by Fuglede [Fu], by using a different notion of asymmetry. However, the study of stability was carried on in full generality by Hall-Hayman-Weitsman [HHW] and Hall [Ha]. The results proved in these two papers combine together to show

$$A(E) \leq C(n)\delta(E)^{1/4}. \tag{3.4}$$

The approach used for the proof of this result relies on symmetrization theory, more precisely on the use of Schwarz symmetrization. In [Ha] it is shown that whenever E is an axially symmetric set (which can be obtained by means of Schwarz symmetrization), then

$$A(E) \leq C(n)\delta(E)^{1/2}, \quad \text{for } E \text{ axially symmetric.} \tag{3.5}$$

In considering axially symmetric sets, we are in practice reducing the problem to one-dimensional objects, and therefore it is much easier to prove an estimate like (3.5) than in a full general setting. The issue of reducing the problem to the class of axially symmetric sets is dealt with in [HHW], where such a reduction step is achieved: for an arbitrary set E it is possible to find a direction ν such that, if E^* is the set obtained by applying Schwarz symmetrization to E, then

$$A(E) \leq C(n)A(E^*)^{1/2}, \quad \delta(E^*) \leq \delta(E). \tag{3.6}$$

We will later see that, while the second inequality in (3.6) follows immediately from Schwartz symmetrization, the first one requires more attention. By combining these two results, one obtains (3.4). However, this approach does not yield and optimal exponent in the decay of the deficit, since in reducing E to an axially symmetric set there is some loss of control. It was actually conjectured by Hall that the same result held with the sharp exponent $1/2$ of $\delta(E)$. The theorem we will prove, that was originally proved in [FMP], states precisely this.

Theorem 3.3 (Stability of the isoperimetric inequality)**.** *For $n \geq 2$ there exists a constant $C(n)$ such that*

$$A(E) \leq C(n)\delta(E)^{1/2}, \tag{3.7}$$

for any $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$. Equivalently,

$$n\,|B|^{1/n}\,|E|^{(n-1)/n}\left(1 + \frac{A(E)^2}{C(n)}\right) \leq P(E). \tag{3.8}$$

Our approach towards proving Theorem 3.3 will follow that of [Ma2]. The key difference with the previous, non-sharp stability results will be the intermediate reduction step between the general set E and the axially symmetric set E^*. In short, the idea is that in reducing E to E^*, we are making an excessive leap which leads to the loss of control in the sharp decay rate. To prevent this, we introduce an intermediate reduction step into another type of symmetric set.

Apart from this, there are several ideas that play major roles in how we approach the problem. Before briefly explaining them, we remark a few things. First of all, with this analysis we do not keep track of the constant $C(n)$. In particular, the study of the anisotropic version will later reveal that a value we may consider for $C(n)$ is

$$C(n) = \frac{187n^7}{(2 - 2^{n-1/n})^{3/2}}.$$

Second of all, in dimension $n = 1$, (3.7) is trivial: either $\delta(E) = 0$ and $A(E) = 0$, or $\delta(E) \geq 2 \geq A(E)$. As a consequence, we shall always assume $n \geq 2$ in what follows. Finally, (3.7) is sharp in the decay rate of the deficit. This can be checked

by computing the Taylor expansion of the isoperimetric deficit and the asymmetry for the family of axially symmetric ellipsoids

$$E_r = \left\{ x \in \mathbb{R}^n : \ (rx_1)^2 + \sum_{i=2}^{n} x_i^2 = 1 \right\},$$

as $r \to 1$.

One key idea which not only appears in the proof of Theorem 3.3, but also when dealing with the more general anisotropic problem, is to consider *almost optimal sets*, i.e., sets with small isoperimetric deficit. The other recurring idea is to try to reduce the problem to a smaller class of sets, where showing (3.7) is hopefully simpler. Often, these reduction steps amount to showing that given a set E we can find another set F with some good additional properties, like symmetries, boundedness, etc. such that the reduction inequalities

$$A(E) \leq C(n) \left(A(F) + \delta(E)^{1/2} \right), \quad \delta(F) \leq C(n)\delta(E),$$

hold true. The second inequality behaves as expected given that F has better properties than E, while the first one goes in the opposite direction, especially under the small deficit assumption. Hence both are in competition.

One last fact that we may remark is that the infimum in (3.2) is always attained. Indeed, for any set E of finite volume in $\mathbb{R}^n$, we may define the function $f(x) = |E\Delta B(x,r)|/|E|$, for $x \in \mathbb{R}^n$, given that $|B(x,r)| = |E|$. Then computing $A(E)$ amounts to finding the infimum of $f(x)$ over $x \in \mathbb{R}^n$. Notice that f is bounded by 0 and 2. Therefore, if f does not attain a minimum, we can find $\{x_h\}_{h=1}^{\infty}$ with $|x_h| \to \infty$ as $h \to \infty$ and such that $f(x_h)$ strictly decreases. Up to a subsequence, we can also assume $|x_h - x_k| > 2r$ for $k \neq h$, which grants $B(x_h, r) \cap B(x_k, r) = \emptyset$ whenever $k \neq h$. Hence,

$$|E| \geq \sum_{h=1}^{\infty} |B(x_h, r) \cap E| ,$$

making $|B(x_h, r) \cap E| \to 0$ as $h \to \infty$. This, however, contradicts the fact that $f(x_h)$ decreases strictly, since

$$\begin{aligned} f(x_h) &= \frac{|E\Delta B(x_h, r)|}{|E|} = \frac{|E \cup B(x_h, r)| - |E \cap B(x_h, r)|}{|E|} \\ &= \frac{|E| + |B(x_h, r)| - 2\,|E \cap B(x_h, r)|}{|E|}, \end{aligned}$$

and by taking the limit as $h \to \infty$ we obtain $f(x_h) \to 2$. Therefore given any set E with finite volume, we may always find a ball that minimizes $A(E)$.

3.1 Small isoperimetric deficit

Let us fix a constant $\delta(n)$. Suppose $K(n)$ is another constant for which

$$A(E) \leq K(n)\delta(E)^{1/2}, \quad \text{whenever } \delta(E) \leq \delta(n).$$

If $\delta(E) > \delta(n)$, then as $A(E) \leq 2$, we can estimate

$$A(E) \leq 2\frac{1}{\delta(n)^{1/2}}\delta(E)^{1/2},$$

and therefore we can set $C(n) = \max\left\{K(n), 2\delta(n)^{-1/2}\right\}$ to obtain (3.7). Hence, we can always assume that E is almost optimal.

3.1.1 Reduction to bounded sets

The first step towards proving Theorem 3.3 will be to reduce E to a bounded set, with a bound that only depends on the dimension. To this end, we present a lemma that turns out to be very useful for proving reduction inequalities.

Lemma 3.4. *There exists constants $C(n)$ and $\varepsilon(n)$ such that, when E and F are sets of finite perimeter in $\mathbb{R}^n$ with*

$$|E\Delta F| \leq \varepsilon |E|, \quad P(F) \leq P(E) + \varepsilon |E|^{1/n'}, \tag{3.9}$$

for $\varepsilon \leq \varepsilon(n)$, then

$$A(E) \leq A(F) + 4\varepsilon, \quad \delta(F) \leq \delta(E) + C(n)\varepsilon.$$

Proof. Let B_F be the ball for which $A(F)$ attains its infimum, and let B_E be a ball with the same center as B_F and measure equal to $|E|$ (that is, B_E is a competitor in $A(E)$). Then,

$$\begin{aligned} A(E)|E| &\leq |E\Delta B_E| \leq |E\Delta F| + |F\Delta B_F| + |B_F\Delta B_E| \\ &\leq \varepsilon|E| + A(F)|F| + ||E| - |F|| \leq 2\varepsilon|E| + A(F)|F| \end{aligned}$$

where we used two facts. The first one being that either $B_E \subset B_F$ or $B_F \subset B_E$, which implies $|B_F\Delta B_E| = ||E| - |F||$ since $|E| = |B_E|$ and $|F| = |B_F|$, and the second one being that $||E| - |F|| \leq |E\Delta F| \leq \varepsilon|E|$. Now, since $|F| \leq |E| + |E\Delta F|$, and $A(F) \leq 2$, we find

$$A(F)|F| \leq A(F)|E| + A(F)|E\Delta F| \leq A(F)|E| + 2\varepsilon|E|,$$

which, together with the previous estimate shows $A(E) \leq A(F) + 4\varepsilon$. It remains to show that $\delta(F) \leq \delta(E) + C(n)\varepsilon$. For this, we can use the definition of $\delta(E)$ and the taylor expansion of $(1-\varepsilon)^{-1/n'}$ to show that

$$\begin{aligned} \delta(E) - \delta(F) &\leq C(n)\left[\frac{P(E)}{|E|^{1/n'}}\left((1-\varepsilon)^{-1/n'} - 1\right) + \varepsilon(1-\varepsilon)^{-1/n'}\right] \\ &\leq C(n)\varepsilon\left[\frac{P(E)}{|E|^{1/n'}} + 1\right]. \end{aligned}$$

□

Theorem 3.5. *There exist constants $\delta(n), C(n)$ and $L(n)$ such that for every set E with finite volume and $\delta(E) \leq \delta(n)$ we can find a set F such that $F \subset [0, |E|^{1/n} L(n)]^n$ and*

$$A(E) \leq C(n)\left(A(F) + \delta(F)\right), \quad \delta(F) \leq C(n)\delta(E). \tag{3.10}$$

The idea for how the set F is constructed is to take an appropriate cutoff of E in each direction of the canonical basis of $\mathbb{R}^n$. The small deficit assumption makes it impossible for E to accumulate too much volume too far away, and F must have similar volume and perimeter, while at the same time being confined within a cube.

Proof. Given E with $\delta(E) \leq \delta(n)$, it is enough to show that there exists a set $F \subset [0, |E|^{1/n} L(n)] \times \mathbb{R}^{n-1}$ with

$$|E \Delta F| \leq C(n)\delta(E)\,|E|\,, \quad P(F) \leq P(E) + C(n)\delta(E)\,|E|^{1/n'}\,,$$

since then we can use Lemma 3.4 to obtain (3.10), and by iterating this argument we complete the proof.

By Theorem 1.11 we can assume that E is bounded with polyhedral boundary, and that

$$\mathcal{H}^{n-1}\left(\{x \in \mathcal{F}E :\ \nu_E(x) = \pm e_1\}\right) = 0. \tag{3.11}$$

This is, since $\mathcal{F}E$ is never normal to e_1, the section of E sliced by the hyperplane $\{x_1 = t\}$ is of $\mathcal{H}^{n-1}$-measure zero. This implies that the function $v(t) = \mathcal{H}^{n-1}(E \cap \{x_1 = t\})$ is continuous, piecewise affine, and with compact support $[a, b] \subset \mathbb{R}^n$ (see Figure 4). Furthermore,

$$v(t) = |E|\,\gamma'(t), \quad \text{a.e. } t \in (a, b),$$

where

$$\gamma(t) := \frac{E \cap \{x_1 < t\}}{|E|}.$$

Since E has polyhedral boundary and (3.11) is in force, this can be checked by expressing E as a graph around $\{x_1 = t\}$ and using the area formula [Ma1, Theorem 8.1]. With the assumption that $\delta(E) \leq \delta(n)$, we can provide a lower bound on $v(t)$ in terms of $\gamma(t)$. Indeed, thanks to (3.11) we have $P(E) = P(E|\,\{x_1 > t\}) + P(E|\,\{x_1 < t\})$, with

$$\begin{aligned} P(E \cap \{x_1 < t\}) &= v(t) + P(E|\,\{x_1 < t\}), \\ P(E \cap \{x_1 > t\}) &= v(t) + P(E|\,\{x_1 > t\}). \end{aligned}$$

Combining this with the isoperimetric inequality and the definition of γ,

$$2v(t) \geq n\,|B|^{1/n}\,|E|^{1/n'}\left(\gamma(t)^{1/n'} + (1 - \gamma(t))^{1/n'}\right) - P(E),$$

which can be rewritten in the form

$$v(t) \geq \frac{n\,|B|^{1/n}\,|E|^{1/n'}}{2}\left[\Psi(\gamma(t)) - \delta(E)\right], \tag{3.12}$$

where $\Psi : [0, 1] \longrightarrow [0, 1]$ is the strictly concave function defined by

$$\Psi(\gamma) = \gamma^{1/n'} + (1 - \gamma)^{1/n'} - 1. \tag{3.13}$$

The function Ψ takes its maximum value $2(2^{1/n} - 1)$ when $\gamma = 1/2$, and it is equal to zero at $\gamma = 0, 1$ (see Figure 9). Therefore we obtain the lower bound

$$\Psi(\gamma) \geq 2(2^{1/n} - 1)\min\{\gamma, 1 - \gamma\}\,. \tag{3.14}$$

Now, assuming $\delta(n) < \Psi(1/2)$, we can find $(t_1, t_2) \subset (a, b)$ such that

$$\begin{aligned} \gamma(t_1) &= 1 - \gamma(t_2), \\ 2\delta(E) &= \Psi(\gamma(t_1)) = \Psi(\gamma(t_2)), \\ \Psi(\gamma(t)) &\geq 2\delta(E) \quad \forall t \in (t_1, t_2). \end{aligned}$$

Notice that given any t_1, the first condition forces the value of t_2, and the second one forces the value of t_1. The third one is given by the convexity of Ψ. As a consequence of these, we can use (3.12) to establish the following lower bounds for every $t \in (t_1, t_2)$,

$$v(t) \geq \frac{n\,|B|^{1/n}\,|E|^{1/n'}\,\delta(E)}{2}, \tag{3.15}$$

$$v(t) \geq \frac{n\,|B|^{1/n}\,|E|^{1/n'}\,\Psi(\gamma(t))}{4}. \tag{3.16}$$

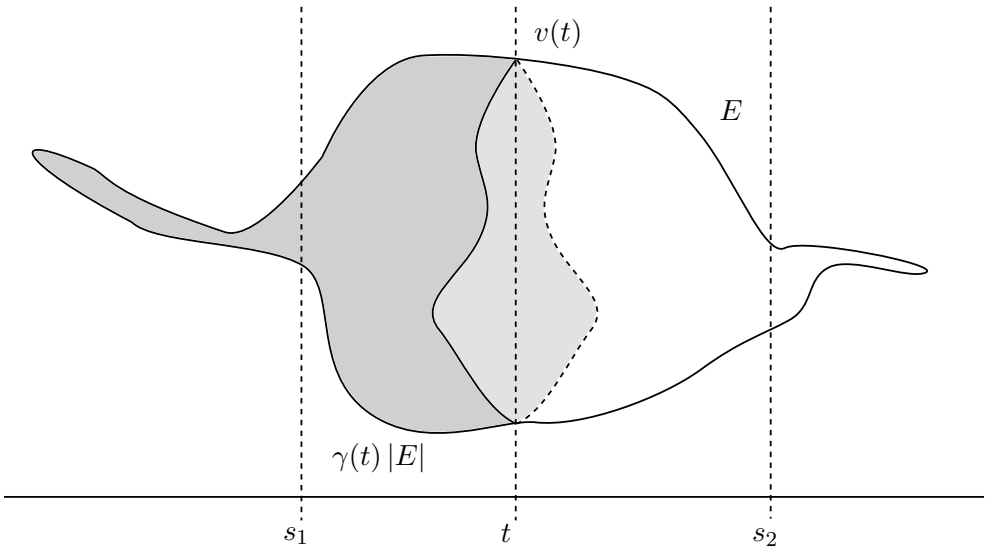


FIGURE 2: If E is almost optimal, for $s_1 < t < s_2$ the volume of $E \cap \{x_1 < t\}$ is not too small or too big compared to $|E|$. We expect a lower bound for $v(t)$ in terms of $\delta(E)$.

We can in fact extend (3.15) to a larger interval in order to find the appropriate cutoff to define F. Let (s_1, s_2) be the maximal interval for which (3.15) holds,

$$s_1 := \max\left\{ t \in (a, t_1] :\ v(t) \leq \frac{n\,|B|^{1/n}\,|E|^{1/n'}\,\delta(E)}{2} \right\},$$

$$s_2 := \min\left\{ t \in [t_2, b) :\ v(t) \leq \frac{n\,|B|^{1/n}\,|E|^{1/n'}\,\delta(E)}{2} \right\}.$$

Now we claim that $s_2 - s_1 \leq L(n)$. Indeed, by (3.14),

$$\gamma(t_1) \leq \frac{\Psi(\gamma(t_1))}{2(2^{1/n} - 1)} = \frac{\Psi(\gamma(t_1))/2}{2^{1/n} - 1} = \frac{\delta(E)}{2^{1/n} - 1}.$$

We notice that since $v(t) = |E|\,\gamma'(t)$, we have $\int_{s_1}^{t_1} v(t)\,\mathrm{d}t = |E|\,(\gamma(t_1) - \gamma(s_1)) \leq |E|\,\gamma(t_1)$, and therefore, by (3.15) on $(s_1, t_1]$,

$$t_1 - s_1 \leq \frac{2}{n\,|B|^{1/n}\,|E|^{1/n'}\,\delta(E)} \int_{s_1}^{t_1} v(t)\,\mathrm{d}t = \frac{2\,|E|^{1/n}\,\gamma(t_1)}{n\,|B|^{1/n}\,\delta(E)} \leq \frac{|E|^{1/n}\,L(n)}{3}.$$

By the same argument we obtain the estimate $s_2 - t_2 \leq |E|^{1/n} L(n)/3$, and now using (3.16) we bound $t_2 - t_1$,

$$\begin{aligned} t_2 - t_1 &\leq \frac{4}{n\,|B|^{1/n}\,|E|^{1/n'}} \int_{t_1}^{t_2} \frac{v(t)\,\mathrm{d}t}{\Psi(\gamma(t))} = \frac{4\,|E|^{1/n}}{n\,|B|^{1/n}} \int_{t_1}^{t_2} \frac{\gamma'(t)\,\mathrm{d}t}{\Psi(\gamma(t))} \\ &= \frac{4\,|E|^{1/n}}{n\,|B|^{1/n}} \int_0^1 \frac{\mathrm{d}\gamma}{\Psi(\gamma)} \leq \frac{|E|^{1/n}\,L(n)}{3} \end{aligned}$$

since $\int_0^1 \frac{1}{\psi(\gamma)}\,\mathrm{d}\gamma$ is a constant that depends only on the dimension n. We end up with

$$s_2 - s_1 = (s_2 - t_2) + (t_2 - t_1) + (t_1 - s_1) \leq |E|^{1/n}\,L(n),$$

which completes the proof of the claim.

Let us define the set F as

$$F = E \cap \{x \in \mathbb{R}^n : s_1 < x_1 < s_2\} - s_1 e_1.$$

It is left to show that F satisfies our requirements in order to complete the proof. Indeed, it is contained in $[0, |E|^{1/n}\,L(n)] \times \mathbb{R}^{n-1}$ and it has

$$|E \Delta F| \leq |E|\,[\gamma(t_1) + (1 - \gamma(t_2))] \leq C(n)\,|E|\,\delta(E).$$

By the continuity of v and the definition of s_1 and s_2 we have $v(s_1) = v(s_2) = C(n)\delta(E)\,|E|^{1/n'}$, and thus

$$\begin{aligned} P(F) &= P(E|(s_1, s_2) \times \mathbb{R}^{n-1}) + \sum_{i=1}^{2} \mathcal{H}^{n-1}\left(E \cap \{x_1 = s_i\}\right) \\ &\leq P(E) + v(s_1) + v(s_2) \leq P(E) + C(n)\delta(E)\,|E|^{1/n'}. \end{aligned}$$

As remarked, simple iteration of this argument for the directions of $e_2, ..., e_n$ yields the desired result. □

3.1.2 A mild quantitative isoperimetric inequality

The following theorem follows as an immediate corollary of Theorem 3.5.

Theorem 3.6. *For every $\varepsilon > 0$ there exists $\delta(n) > 0$ such that $A(E) \leq \varepsilon$ whenever $\delta(E) \leq \delta(n)$.*

Proof. The proof is by contradiction. Let $\{E_h\}$ be a sequence such that $\delta(E_h) \to 0$ while $\lim_{h\to\infty} A(E_h) = \varepsilon > 0$. We may assume $|E_h| = |B|$. Now define F_h to be the associated set to E_h that is given by Theorem 3.5. If $F_h \to F$, then $\delta(F) = 0$, and $A(F) = \varepsilon > 0$ by the lower semicontinuity of the deficit and by the continuity of the Fraenkel asymmetry, and this contradicts the equality case in the isoperimetric inequality. □

During the proof we used the fact that if $|E_h| = |E|$ and $|E_h \Delta E| \to 0$ as $h \to \infty$, then $A(E_h) \to A(E)$. This can be easily checked by using the optimal ball for $A(E)$ for estimating $A(E_h)$, and introducing E as a comparison set. The same can be done to estimate $A(E)$, reaching

$$A(E) - \frac{|E\Delta E_h|}{|E|} \leq A(E_h) \leq A(E) + \frac{|E\Delta E_h|}{|E|}.$$

Remark. This proof relies on the characterization of the equality case in Theorem 2.1, which was proven through the use of the Steiner inequality (Theorem 2.2). Symmetry is needed in the problem if one aims to use this technique, and therefore we do not expect to find a analogous result in the anisotropic problem. In fact, the characterization of equality cases in that case follows from the stability result (Theorem 5.3), although it can be deduced by varying certain arguments in the proof of Theorem 4.2 and via a finer analysis of the Brenier map [FiMP, Appendix A]. This shows again how the Euclidean case can be attacked exploiting its symmetry, in contrast to the anisotropic case.

3.1.3 Reduction to indecomposable sets

In this part we will show how the small deficit assumption allows us to reduce E to an indecomposable set.

Definition 3.7. A set of finite perimeter E in $\mathbb{R}^n$ with $0 < |E| < \infty$ is said to be *indecomposable* if whenever $E = E_1 \cup E_2$, with $E_1 \cap E_2 = \emptyset$ and $P(E) = P(E_1) + P(E_2)$ it holds that $\min\{|E_1|, |E_2|\} = 0$.

Indecomposability is the equivalent to *connectedness* in the theory of sets of finite perimeter [Ma1, Exercise 16.9]. In a similar way to how every open set with finite measure can be decomposed into the union of countably many disjoint open connected components, every set of finite perimeter E with $0 < |E| < \infty$ can be decomposed as

$$E = \bigcup_{h \in \mathbb{N}} E_h,$$

where each E_h is indecomposable and $E_h \cap E_k = \emptyset$ whenever $h \neq k$ [AFP, Example 4.18]. Additionally,

$$P(E) = \sum_{h \in \mathbb{N}} P(E_h).$$

One way to motivate why it is possible to reduce the problem to indecomposable sets is to look at optimal sets in the isoperimetric inequality. If E is optimal, then it must be indecomposable: if $E = E_1 \cup E_2$, with $E_1 \cap E_2 = \emptyset$ and $P(E) = P(E_1) + P(E_2)$, then

$$P(E) = P(E_1) + P(E_2) \geq n\,|B|^{1/n} \left(|E_1|^{1/n'} + |E_2|^{1/n'} \right) \geq n\,|B|^{1/n}\,|E|^{1/n'},$$

and the inequality is strict unless either $|E_1| = 0$ or $|E_2| = 0$.

If E decomposes into $\{E_h\}_{h \in \mathbb{N}}$ with decreasing volume, it would reasonable to expect some control of $|E \setminus E_1|$ by $\delta(E)$ provided $|E_1|$ was large enough: we could hope to apply Lemma 3.4 with $F = E_1$. Otherwise, if two or more E_h gather a large part of $|E|$, then we expect $\delta(E)$ to be large, since E would be far away from being optimal.

Theorem 3.8. *There exist $C(n)$ and $\delta(n)$ such that, if $\delta(E) \leq \delta(n)$, then an indecomposable set F can be found such that*

$$A(E) \leq A(F) + C(n)\delta(E), \quad \delta(F) \leq C(n)\delta(E).$$

Proof. Let $E = \bigcup_{h\in\mathbb{N}} E_h$ be the decomposition of E into indecomposable sets of decreasing volume, and assume $|E| = |B|$. By concavity,

$$P(E) = \sum_{h\in\mathbb{N}} P(E_h) \geq |B|^{1/n} \sum_{h\in\mathbb{N}} |E_h|^{1/n'} \geq n\,|B|^{1/n}\,|E|^{1/n'}.$$

In terms of the deficit, this is

$$n\,|B|\,\delta(E) \geq \left[\sum_{h\in\mathbb{N}} \left(\frac{|E_h|}{|E|}\right)^{1/n'}\right] - 1.$$

Step 1. Assume $|E_1| \geq |E|/2$. Then, by concavity of $t \mapsto t^{1/n'} + (1-t)^{1/n'} - 1$ and since its quotient with $t \mapsto (1-t)^{1/n'}$ is a positive constant as $t \to 1$, we find that, for a suitable constant $c_0 = c_0(n)$,

$$n\,|B|\,\delta(E) \geq \left(\frac{|E_h|}{|E|}\right)^{1/n'} + \left[1 - \frac{|E_h|}{|E|}\right]^{1/n'} \geq c_0 \left(1 - \frac{|E_h|}{|E|}\right)^{1/n'},$$

which means that $|E \setminus E_1| \leq C(n)\delta(E)^{n'}$. Since $P(E_1) \leq P(E)$ we conclude by taking $F = E_1$ and using Lemma 3.4 .

Step 2. Assume now that $|E_1| < |E|/2$. Then we aim to bound $\delta(E)$ from below by a positive constant depending on n. Since $|E_1|/|E| < 1/2$, we can find $M \geq 2$ such that

$$\sum_{h<M} \frac{|E_h|}{|E|} < \frac{1}{1}, \quad \sum_{h\leq M} \frac{|E_h|}{|E|} \geq 1/2.$$

Again by concavity,

$$n\,|B|\,\delta(E) \geq \left(\sum_{h<M} \frac{|E_h|}{|E|}\right)^{1/n'} + \left(\sum_{h\geq M} \frac{|E_h|}{|E|}\right)^{1/n'} - 1 \geq c_0 \left(\sum_{h<M} \frac{|E_h|}{|E|}\right)^{1/n'},$$

$$n\,|B|\,\delta(E) \geq \left(\sum_{h\leq M} \frac{|E_h|}{|E|}\right)^{1/n'} + \left(\sum_{h>M} \frac{|E_h|}{|E|}\right)^{1/n'} - 1 \geq c_0 \left(\sum_{h<M} \frac{|E_h|}{|E|}\right)^{1/n'}.$$

We find

$$C(n)\delta(E)^{n'} \geq \sum_{h<M} \frac{|E_h|}{|E|} + \sum_{h>M} \frac{|E_h|}{|E|} = 1 - \frac{|E_M|}{|E|} \geq 1 - \frac{|E_1|}{|E|} \geq \frac{1}{2}.$$

□

3.2 Reduction to symmetric sets

Upon recalling the discussion preceding Theorem 3.3, we remark that the way we approach the main reduction result is by adding an intermediate step between general sets and axially symmetric sets. In order to understand this key idea, we first need to introduce a couple of notions of symmetry and symmetrization techniques.

3.2.1 Schwarz symmetrization

In order to reduce the problem to axially symmetric sets we will need to use *Schwarz symmetrization*. Given a direction $\nu \in \mathbb{S}^{n-1}$ and a measurable set E, the function

$$\begin{aligned} v : \mathbb{R} &\longrightarrow [0, \infty) \\ t &\longmapsto \mathcal{H}^{n-1}\left(E \cap \{x \in \mathbb{R}^n : \ x \cdot \nu = t\}\right) \end{aligned}$$

is measurable by Fubini's Theorem, and if $|E| < \infty$, then it is in $L^1(\mathbb{R})$. Let $r(t)$ be the radius of an $(n-1)$-dimensional ball with $\mathcal{H}^{n-1}$-measure equal to $v(t)$,

$$r(t) = \left(\frac{v(t)}{\mathcal{H}^{n-1}(B \cap \mathbb{R}^{n-1})}\right)^{1/(n-1)}.$$

The we define the *Schwarz symmetral* E^* *of* E as

$$E^* = \{x \in \mathbb{R}^n : \ r\,(x \cdot \nu) > 0, \ |x - (x \cdot \nu)\nu| < r(x \cdot \nu)\}\,.$$

This corresponds to measuring the intersection of E and the orthogonal hyperplane to ν at height t, and centering an $(n-1)$-ball of equal measure around $t\nu$, for each $t \in \mathbb{R}$. By construction, E^* is *axially symmetric* along ν and by Fubini's Theorem, $|E^*| = |E|$. Intuitively, the perimeter of each section is minimized by the isoperimetric inequality, and therefore we expect $P(E^*) \leq P(E)$.

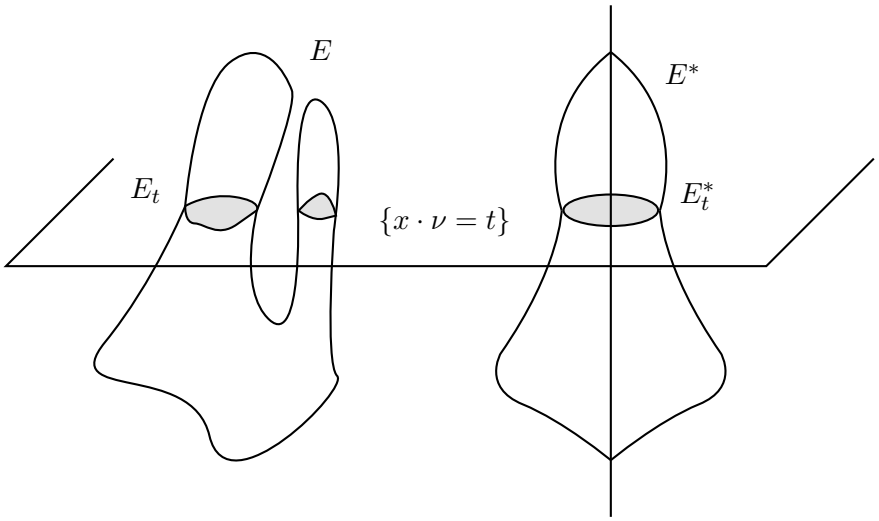


FIGURE 3: Schwarz symmetrization.

Theorem 3.9 (Schwarz inequality)**.** *If E is a set of finite perimeter in $\mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and E^* denotes its Schwarz symmetral with respect to ν, we have*

$$P(E^*) \leq P(E).$$

Before coming to the proof of the Schwarz inequality we need the following lemma, which we will use in the proof of Theorem 3.3 as well.

Lemma 3.10. *If E is a set of finite perimeter in $\mathbb{R}^n$, then the function $v(t) = \mathcal{H}^{n-1}\left(E \cap \{x_1 = t\}\right)$ has bounded variation on $\mathbb{R}$, and if*

$$\mathcal{H}^{n-1}\left(\{x \in \mathcal{F}E : \ \nu_E(x) = \pm e_1\}\right) = 0 \tag{3.17}$$

holds true, then v is absolutely continuous on $\mathbb{R}$, with classical derivative

$$v'(t) = -\int_{\mathcal{F}E_t} \frac{\nu_E \cdot e_1}{\sqrt{1 - |\nu_E \cdot e_1|^2}} \, \mathrm{d}\mathcal{H}^{n-2}. \tag{3.18}$$

Proof. In order to show that v has bounded variation on $\mathbb{R}$ it suffices to show that

$$\sup \left\{ \int_{\mathbb{R}} v(t)\varphi'(t) \, \mathrm{d}t : \ \varphi \in C^1_c(\mathbb{R}; [-1,1]) \right\} < \infty.$$

Let us consider a sequence $\{\psi_h\}_{h\in\mathbb{N}}$, with $\psi_h \in C^1_c(\mathbb{R}^{n-1}; [0,1])$, and ψ_h increasing to 1. Now, define $f_h(x) = \varphi(x_1)\psi_h(x - x_1 e_1)$. Then $f_h \in C^1_c(\mathbb{R}^n)$ and

$$\begin{aligned}\int_{\mathbb{R}} \varphi' v &= \lim_{h\to\infty} \int_{\mathbb{R}} \mathrm{d}t \int_{E_t} \varphi'(t)\psi_h(z) \, \mathrm{d}\mathcal{H}^{n-1}(z) = \lim_{h\to\infty} \int_E \frac{\partial f_h}{\partial x_1}(x) \, \mathrm{d}x \\ &= \lim_{h\to\infty} \int_{\mathcal{F}E} f_h(x)\nu_E(x) \cdot e_1 \, \mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\mathcal{F}E} \varphi(x_1)\nu_E(x) \cdot e_1 \, \mathrm{d}\mathcal{H}^{n-1}(x).\end{aligned}$$

Since $\varphi(x_1) \in [-1,1]$, the right-hand side is bounded by $\mathcal{H}^{n-1}(\mathcal{F}E) = P(E)$, and by the arbitrariness of φ, v has bounded variation.

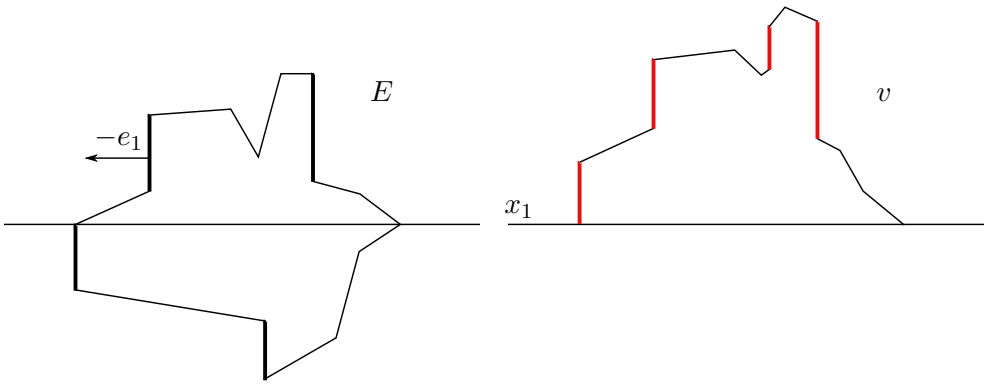


FIGURE 4: The continuity of v is spoiled whenever there are parts of the reduced boundary of E with $\nu_E = \pm e_1$.

If we assume the validity of (3.17), then $\sqrt{1 - |\nu_E \cdot e_1|^2} > 0$ at $\mathcal{H}^{n-1}$-a.e. point of $\mathcal{F}E$. Using the Coarea Formula (Theorem 1.13) we find

$$\begin{aligned}&\int_{\mathcal{F}E} \varphi(x_1)\nu_E(x) \cdot e_1 \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ &\qquad = \int_{\mathcal{F}E} \varphi(x_1) \frac{\nu_E(x) \cdot e_1}{\sqrt{1 - |\nu_E(x) \cdot e_1|^2}} \sqrt{1 - |\nu_E(x) \cdot e_1|^2} \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ &\qquad = \int_{\mathbb{R}} \varphi(t) \, \mathrm{d}t \int_{\mathcal{F}E_t} \frac{\nu_E \cdot e_1}{\sqrt{1 - |\nu_E \cdot e_1|^2}} \, \mathrm{d}\mathcal{H}^{n-2},\end{aligned}$$

from which (3.18) follows. □

Now we prove the Schwartz inequality.

Proof of Theorem 3.9. Without loss of generality we may assume that E has finite perimeter and consider its Schwartz symmetral E^* with respect to the x_1-dimension. As $\mathcal{H}^{n-1}(\mathcal{F}E) < \infty$, it may be

$$\mathcal{H}^{n-1}(\{x \in \mathcal{F}E : \nu_E(x) = \pm\nu\}) > 0$$

for at most countably many $\nu \in \mathbb{S}^{n-1}$. Therefore, up to a rotation of E we may also assume (3.17). In particular, $\sqrt{1-|\nu_E \cdot e_1|^2} > 0$ on $\mathcal{F}E$ up to a $\mathcal{H}^{n-1}$-null set. Therefore, by the Coarea Formula (Theorem 1.13), the Jensen inequality applied to $\Phi(z) := \sqrt{1+z^2}$ for $z \in \mathbb{R}$, and by (3.18) in the proof of Lemma 3.10 above, we find

$$\begin{aligned}
P(E) = \mathcal{H}^{n-1}(\mathcal{F}E) &= \int_{\mathcal{F}E} \frac{\sqrt{1-|\nu_E \cdot e_1|^2}}{\sqrt{1-|\nu_E \cdot e_1|^2}} = \int_{\mathbb{R}} \mathrm{d}t \int_{\mathcal{F}E_t} \frac{1}{\sqrt{1-|\nu_E \cdot e_1|^2}}\, \mathrm{d}\mathcal{H}^{n-2} \\
&= \int_{\mathbb{R}} \int_{\mathcal{F}E_t} \Phi\left(\frac{\nu_E \cdot e_1}{\sqrt{1-|\nu_E \cdot e_1|^2}}\right) \mathrm{d}\mathcal{H}^{n-2} \\
&\geq \int_{\mathbb{R}} \mathcal{H}^{n-2}(\mathcal{F}E_t)\, \Phi\left(\frac{1}{\mathcal{H}^{n-2}(\mathcal{F}E_t)} \int_{\mathcal{F}E_t} \frac{\nu_E \cdot e_1}{\sqrt{1-|\nu_E \cdot e_1|^2}}\right) \mathrm{d}t \\
&= \int_{\mathbb{R}} \sqrt{\mathcal{H}^{n-2}(\mathcal{F}E_t)^2 + v'(t)^2}\, \mathrm{d}t.
\end{aligned}$$

Therefore, we have proved the following lower bound for the perimeter of E under the geometric assumption (3.17),

$$P(E) \geq \int_{\mathbb{R}} \sqrt{\mathcal{H}^{n-2}(\mathcal{F}E_t)^2 + v'(t)^2}\, \mathrm{d}t. \tag{3.19}$$

The same argument applies to E^* (in particular, (3.17) holds for E^* as well), and in fact in this case equality holds, as $\nu_{E^*} \cdot e_1$ has to be constant on $\mathcal{F}E_t^*$ thanks to the symmetry of E^*. Therefore,

$$P(E^*) = \int_{\mathbb{R}} \sqrt{\mathcal{H}^{n-2}(\mathcal{F}E_t^*)^2 + v'(t)^2}\, \mathrm{d}t. \tag{3.20}$$

Now recall that $(E^*)_t$ is, by definition, an $(n-1)$-dimensional ball with the same $\mathcal{H}^{n-1}$-measure as E_t. Using the isoperimetric inequality in $\mathbb{R}^{n-1}$,

$$\mathcal{H}^{n-2}(\mathcal{F}E_t) \geq \mathcal{H}^{n-2}(\mathcal{F}E_t^*).$$

We conclude that $P(E) \geq P(E^*)$. □

3.2.2 Reflections by symmetry and n-symmetric sets

We now introduce the notion of n-symmetry, which we will use in proving Theorem 3.3 as the intermediate reduction step between generic sets and axially symmetric sets.

Definition 3.11 (n-symmetry)**.** A set $E \subset \mathbb{R}^n$ is said to be *n-symmetric* if there are n-orthogonal hyperplanes such that E is invariant under reflection with respect to each of them. The intersection of the n orthogonal hyperplanes of symmetry of E is called the *center of symmetry of E*.

Notice that E is n-symmetric if, up to a rigid motion, it is symmetric with respect to the coordinate hyperplanes. Before giving the symmetrization technique that allows the reduction to n-symmetric sets, we give two lemmas that we will use repeatedly in the following proofs, hinting at why it is relevant to consider n-symmetric sets. As a matter of fact, for any given set E there is no way to immediately know anything about the center x of the ball minimizing $A(E)$. With n-symmetric sets this choice becomes simple: its center of symmetry. This may not coincide with x, but we can use it to estimate $A(E)$ up to a constant.

Lemma 3.12. *Let E be n-symmetric, with center of symmetry at $y \in \mathbb{R}^n$. Then*

$$A(E) \leq \frac{|E\Delta B(y,r)|}{|E|} \leq 3A(E), \quad r = \left(\frac{|E|}{|B|}\right)^{1/n}. \tag{3.21}$$

Proof. Without loss of generality we can assume $|E| = 1$. Let x be such that $A(E) = |E\Delta B(x,r)|$, and let z be the reflection of x with respect to y. Since E is n-symmetric, z is optimal for $A(E)$ as well, meaning that $A(E) = |E\Delta B(z,r)|$. Since y lies in the segment joining x and z, it is clear that $|B(x,r)\Delta B(y,r)| \leq |B(x,r)\Delta B(z,r)|$, and

$$\begin{aligned} A(E) &\leq |E\Delta B(y,r)| \leq |E\Delta B(x,r)| + |B(x,r)\Delta B(y,r)| \\ &\leq |E\Delta B(x,r)| + |B(x,r)\Delta B(z,r)| \\ &\leq 2\,|E\Delta B(x,r)| + |E\Delta B(z,r)| = 3A(E). \end{aligned}$$

□

This argument also proves a more general property. For $k \in \{1, ..., n\}$, we say E is k-symmetric if it is symmetric by reflection with respect to k orthogonal hyperplanes. Using the argument above we have the following lemma.

Lemma 3.13. *Let E be a k-symmetric set, and let H be the $(n-k)$-dimensional affine subspace of $\mathbb{R}^n$ at which the hyperplanes of symmetry of E intersect. Then, recalling* (3.3) *in Definition 3.2,*

$$A(E) \leq A(E|H) \leq 3A(E). \tag{3.22}$$

3.2.3 Symmetrization by reflection

We will now provide a way to reduce a set E in the isoperimetric problem to an n-symmetric one. Given a hyperplane H, we denote by H^+ and H^- the two half-spaces that H divides $\mathbb{R}^n$ into, and define $W : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ to be the reflection with respect to H. Whenever H divides E into two equal parts,

$$\left|E \cap H^+\right| = |E \cap H^-| = \frac{|E|}{2},$$

we define the *symmetrals of E by reflection through H* as

$$\begin{aligned} E^+ &= [E \cap H^+] \cup [W(E \cap H^+)], \\ E^- &= [E \cap H^-] \cup [W(E \cap H^-)]. \end{aligned}$$

Of course, $|E^+| = |E^-| = |E|$, and both E^+ and E^- are symmetric with respect to H. Now we investigate the behavior of the isoperimetric deficit and the Fraenkel asymmetry under symmetrization.

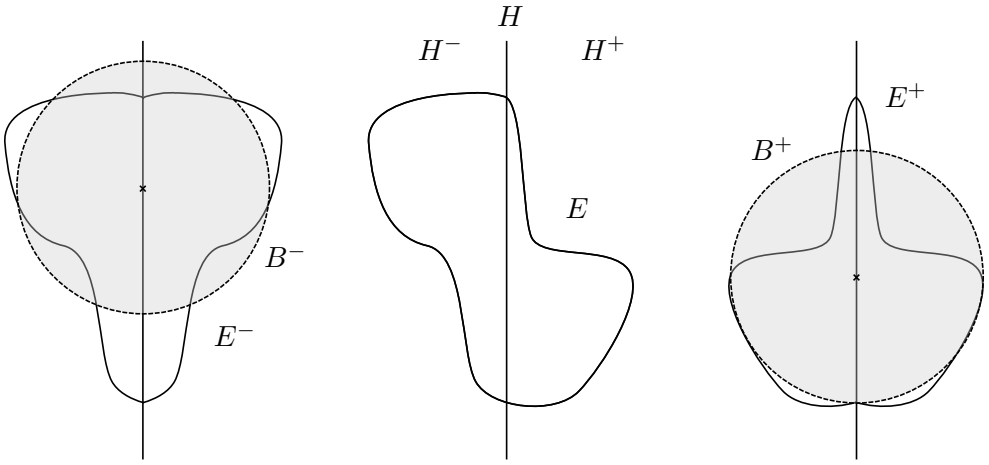


FIGURE 5: Symmetrization by reflection with respect to H. The centers for the balls B^+ and B^- do not necessarily coincide.

First, since

$$P(E) = P(E|H^+) + P(E|H^-) + \mathcal{H}^{n-1}(\mathcal{F}E \cap H) \geq P(E|H^+) + P(E|H^-),$$

and

$$P(E^+) = 2P(E|H^+), \quad P(E^-) = 2P(E|H^-),$$

we find

$$\delta(E) = \frac{P(E)}{n\,|B|^{1/n}\,|E|^{1/n'}} \geq \frac{P(E^+) + P(E^-)}{2n\,|B|^{1/n}\,|E|^{1/n'}} = \frac{\delta(E^+) + \delta(E^-)}{2}. \tag{3.23}$$

We see that the isoperimetric deficit of each reflection is controlled, up to a factor of 2, by that of E. Now we look at the Fraenkel asymmetry. Let B^+ and B^- be the optimal balls for $A(E|H^+)$ and $A(E|H^-)$ respectively,

$$A(E^+|H) = \frac{|E^+ \Delta B^+|}{|E|}, \quad A(E^-|H) = \frac{|E^- \Delta B^-|}{|E|}.$$

Then since B^+ competes in $A(E)$ and both B^+ and B^- are centered at points of H,

$$\begin{aligned} |E|\,A(E) &\leq \left|E\Delta B^+\right| = \left|(E\Delta B^+) \cap H^+\right| + \left|(E\Delta B^+) \cap H^-\right| \\ &\leq \left|(E\Delta B^+) \cap H^+\right| + \left|(E\Delta B^-) \cap H^-\right| + \left|(B^+\Delta B^-) \cap H^-\right| \\ &= \frac{1}{2}\left[|E|\,A(E^+|H) + |E|\,A(E^-|H) + \left|B^+\Delta B^-\right|\right]. \end{aligned}$$

Using this together with Lemma 3.13 yields

$$A(E) \leq 3\left(A(E^+) + A(E^-)\right) + \frac{|B^+\Delta B^-|}{|E|}. \tag{3.24}$$

Note that it is not possible to get rid of the last term in the inequality: it may happen that $A(E^+) = A(E^-) = 0$ while still being $A(E) > 0$, as the following example suggests.

Example 3.14. In $\mathbb{R}^2$, consider the hyperplane $H = \{x_1 = 0\}$ and set

$$E = [B(0,1) \cap \{x_1 < 0\}] \cup [B(3,1) \cap \{x_1 > 0\}].$$

Then it is clear that $A(E) > 0$, while both E^+ and E^- are balls, and therefore $A(E^+) = A(E^-) = 0$.

This shows how it is not enough to choose any arbitrary hyperplane H and then select either E^+ or E^- in order to obtain a reduction to a 1-symmetric set. However, we shall easily overcome this obstacle by simply choosing any orthogonal hyperplane to H as long as it divides E in equal parts.

Theorem 3.15 (n-symmetrization for the isoperimetric inequality)**.** *For every set E there exists an n-symmetric set F such that*

$$A(E) \leq C(n)A(F), \quad \delta(F) \leq C(n)\delta(E). \tag{3.25}$$

Proof. Without loss of generality we may assume that $|E| = |B|$. We can directly consider the case $\delta(E) \leq \delta(n)$ for a given constant $\delta(n)$. Indeed, let E_r be the ellipsoid

$$E_r = \left\{ x \in \mathbb{R}^n : \ (rx_1)^2 + \sum_{i=2}^{n} x_i = 1 \right\}, \quad r < 1,$$

and let $r(n)$ be such that $\delta(E_{r(n)}) = \delta(n)$. Then for every set E with $\delta(E) > \delta(n)$ we choose $F = E_{r(n)}$, which is n-symmetric, and additionaly

$$A(E) \leq 2 = \frac{2}{A(E_{r(n)})} A(F), \quad \delta(F) \leq \delta(n) < \delta(E).$$

From now on we assume $\delta(E) \leq \delta(n)$, $|E| = |B|$ and divide the proof into two steps.

Step 1. We show the existence of two constants $C_0(n)$ and $\delta_0(n)$ such that given E with $|E| = |B|$ and $\delta(E) \leq \delta_0(n)$, and given a pair of mutually orthogonal hyperplanes H_1 and H_2 dividing E into equal parts, then there is a set F, symmetric by reflection with respect to either H_1 or H_2, such that $|E| = |F|$ and the reduction inequalities

$$A(E) \leq C_0(n)A(F), \quad \delta(F) \leq 2\delta(E),$$

hold true.

To prove the claim, we consider the symmetrals by reflections of E generated by H_1 and H_2, namely E_1^+, E_1^-, E_2^+ and E_2^-. By the discussion prior to the theorem, we have $\delta(E_i^\sigma) \leq 2\delta(E)$ for $i = 1, 2$ and $\sigma = +, -$. Let now B_1^+, B_1^-, B_2^+ and B_2^- be the optimal balls for the candidate sets E_i^σ with respect to their hyperplanes of symmetry,

$$A(E_i^{\sigma|H_i}) = \frac{|E_i^\sigma \Delta B_i^\sigma|}{|B|}, \quad i = 1, 2, \quad \sigma = +, -.$$

Then, from (3.24) we know that

$$A(E) \leq 3 \left(A(E_1^+) + A(E_1^-) \right) + \frac{\left| B_1^+ \Delta B_1^- \right|}{|B|},$$

$$A(E) \leq 3\left(A(E_2^+) + A(E_2^-)\right) + \frac{\left|B_2^+ \Delta B_2^-\right|}{|B|}.$$

It suffices to show that, for $\delta(n)$ small enough, one of the two following conditions is satisfied

$$\frac{\left|B_1^+ \Delta B_1^-\right|}{|B|} \leq 17\left[A(E_1^+|H_1) + A(E_1^-|H_1)\right],$$
$$\frac{\left|B_2^+ \Delta B_2^-\right|}{|B|} \leq 17\left[A(E_2^+|H_2) + A(E_2^-|H_2)\right],$$

and to apply Lemma 3.12 in order to conclude the proof of the claim by selecting, respectively, one between E_1^+ and E_1^-, or E_2^+ and E_2^- (the one maximizing the asymmetry). It is precisely here that we solve the problem encountered in Example 3.14.

We will argue by contradiction and assume that both inequalities are false,

$$\left|E_1^+ \Delta B_1^+\right| + \left|E_1^- \Delta B_1^-\right| < \frac{\left|B_1^+ \Delta B_1^-\right|}{17}, \tag{3.26}$$
$$\left|E_2^+ \Delta B_2^+\right| + \left|E_2^- \Delta B_2^-\right| < \frac{\left|B_2^+ \Delta B_2^-\right|}{17}, \tag{3.27}$$

We now look at the two pairs of orthogonally shifted half-balls T_1 and T_2, defined as

$$T_1 = \left(B_1^+ \cap H_1^+\right) \cup \left(B_1^- \cap H_1^-\right),$$
$$T_2 = \left(B_2^+ \cap H_2^+\right) \cup \left(B_2^- \cap H_2^-\right).$$

Using the inequalities (3.26) and (3.27), we have

$$\begin{aligned} |T_1 \Delta T_2| &\leq |T_1 \Delta E| + |T_2 \Delta E| \\ &= \frac{\left|E_1^+ \Delta B_1^+\right|}{2} + \frac{\left|E_1^- \Delta B_1^-\right|}{2} + \frac{\left|E_2^+ \Delta B_2^+\right|}{2} + \frac{\left|E_2^- \Delta B_2^-\right|}{2} \\ &< \frac{1}{17}\frac{\left|B_1^+ \Delta B_1^-\right| + \left|B_2^+ \Delta B_2^-\right|}{2}. \end{aligned} \tag{3.28}$$

If we can now prove that this is now smaller than $1/k\,|T_1 \Delta T_2|$ for some $k > 1$, then we reach a contradiction, finishing the proof of this first step. Here we will look for $k = 17/16$, and we will show that

$$\max\left\{\left|B_1^+ \Delta B_1^-\right|, \left|B_2^+ \Delta B_2^-\right|\right\} \leq 16\,|T_1 \Delta T_2|, \tag{3.29}$$

provided $\delta(E) \leq \delta(n)$. The heuristic reason to expect something like this (see Figure 6) is that $|T_1 \Delta T_2|$ controls the distances by which the half balls T_1 and T_2 are shifted, and in particular it controls $\left|B_1^+ \Delta B_1^-\right|$ and $\left|B_2^+ \Delta B_2^-\right|$. When we are done showing (3.29), plugging it into (3.28) yields

$$|T_1 \Delta T_2| \leq \frac{16}{17}\,|T_1 \Delta T_2|,$$

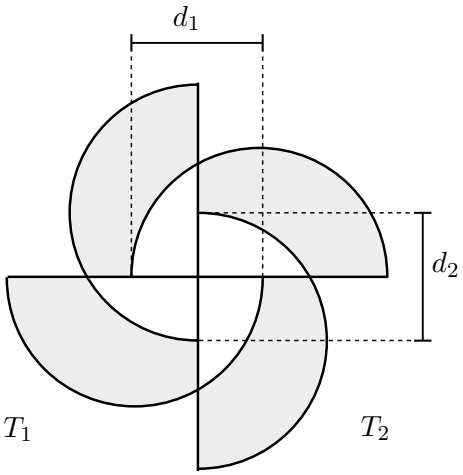


FIGURE 6: The volume of the symmetric difference of T_1 and T_2, the two pairs of shifted half balls, controls the distances d_1 and d_2 between the centers of B_1^+ and B_1^- and of B_2^+ and B_2^-. By making d_1 and d_2 arbitrarily small, each quarter space ends up taking almost as much as the rest.

which implies that $T_1 = T_2$, and $\delta(E) = 0$, against the strict sign in (3.26) and (3.27).

Therefore, it remains to show (3.29). As E_i^σ is symmetric with respect to the hyperplane H_i, we know by Lemma 3.22 that $A(E_i^\sigma|H_i) \leq 3A(E_i^\sigma)$. On the other hand, we also know that $\delta(E_i^\sigma) \leq 2\delta(E)$. Therefore, we can choose $\varepsilon(n)$, and up to taking $\delta_0(n)$ small enough, we can assume

$$\max_{i\in\{1,2\},\sigma\in\{+,-\}} \{A(E), A(E_i^\sigma|H_i)\} \leq \varepsilon(n) \tag{3.30}$$

thanks to Theorem 3.6 applied to E_i^σ. In general, the centers of the balls B_i^σ are all different from the center of the optimal ball for E, and likewise, though each of the H_i's divides E into equal parts, it will happen that none of the four quarter-spaces

$$H_1^\sigma \cap H_2^\tau, \quad \sigma, \tau \in \{+, -\}$$

encloses exactly one quarter of the measure of E or of the balls B_i^σ. However, if $\varepsilon(n)$ is small enough, thanks to (3.30), the centers of the various balls B_i^σ can be chosen as close as wanted to the center of the optimal ball for E, and each quarter-space $H_1^\sigma \cap H_2^\tau$ will therefore be as close as desired to cutting E and each of the balls in four equal parts. Thus, we can choose $\varepsilon(n)$ so small that (3.30) implies

$$|(B_1^\sigma \Delta B_2^\tau) \cap (H_1^\sigma \Delta H_2^\tau)| \geq \frac{|B_1^\sigma \Delta B_2^\tau|}{8}, \tag{3.31}$$

for every $\sigma, \tau \in \{+, -\}$. Using this,

$$|T_1 \Delta T_2| \geq |(T_1 \Delta T_2) \cap (H_1^\sigma \cap H_2^\tau)| = |(B_1^\sigma \Delta B_2^\tau) \cap (H_1^\sigma \cap H_2^\tau)| \geq \frac{|B_1^\sigma \Delta B_2^\tau|}{8},$$

and in particular

$$\left|B_1^+ \Delta B_1^-\right| \leq \left|B_1^+ \Delta B_2^+\right| + \left|B_2^+ \Delta B_1^-\right| \leq 16\,|T_1 \Delta T_2|\,.$$

Since we have similarly $\left|B_2^+ \Delta B_2^-\right| \leq 16\,|T_1 \Delta T_2|$, (3.29) follows, and the first step is proved.

Step 2. We now prove that given any $k \in \{0, ..., n-2\}$ and a k-symmetric set F_k with $\delta(F_k) \leq \delta_0(n)$, then a $k+1$-symmetric set F_{k+1} can be found such that $|F_{k+1}| = |F_k|$ and

$$A(F_k) \leq C(n) A(F_{k+1}), \quad \delta(F_{k+1}) \leq 2\delta(F_k).$$

Indeed, as $k \leq n-2$, we can find two mutually orthogonal hyperplanes H_1 and H_2, both dividing F_k into equal parts and both orthogonal to the k-hyperplanes of symmetry of F_k. On arguing on $(F_k)_1^+, (F_k)_1^-, (F_k)_2^+$ and $(F_k)_2^-$ as in *Step 1*, we have that, by selecting F_{k+1} as one of those four sets,

$$A(F_k) \leq C(n) A(F_{k+1}), \quad \delta(F_{k+1}) \leq 2\delta(F_k).$$

Furthermore, by construction F_{k+1} is $(k+1)$-symmetric.

By a finite induction argument it is immediately shown that for every E such that $\delta(E) \leq 2^{-n}\delta_0(n)$ we can find an $(n-1)$-symmetric set G such that $|E| = |G|$ and

$$A(E) \leq C(n) A(G), \quad \delta(G) \leq 2^{n-1}\delta(E).$$

Now we aim to do a final symmetrization of G, and we cannot apply the same argument as before since we run out of orthogonal hyperplanes. Let H be a hyperplane dividing G into equal parts, and let it be orthogonal to the $(n-1)$-hyperplanes of symmetry of G. Let us consider the symmetrals by reflection G^+ and G^- of G with respect to H. By construction they are both n-symmetric, with $\delta(G^\sigma) \leq 2\delta(G) \leq 2^n\delta(E)$. Furthermore, G^+ and G^- have the same center of symmetry, so that if B_0 is the ball of measure $|E|$ centered at that point, we have

$$\frac{|G^+ \Delta B_0|}{|E|} \leq 3A(G^+), \quad \frac{|G^- \Delta B_0|}{|E|} \leq 3A(G^-),$$

by Lemma 3.24. Therefore,

$$\begin{aligned} A(E) \leq C(n) A(G) &\leq C(n) \frac{|G \Delta B_0|}{|G|} = C(n) \left(\frac{|G^+ \Delta B_0| + |G^- \Delta B_0|}{2\,|E|} \right) \\ &\leq C(n) \left[A(G^+) + A(G^-) \right]. \end{aligned}$$

It suffices to take F as G^+ or G^-, whichever one has greater asymmetry. □

3.3 Proof of Theorem 3.3

We finally come to the proof of Theorem 3.3.

Proof of Theorem 3.3. Without loss of generality we may assume that $|E| = |B|$ and that $\delta(E) \leq \delta(n)$. We now make use of the reduction results we proved earlier. First,

we can assume E to be indecomposable and contained in $[0, L(n)]^n$ by Theorems 3.5 and 3.8. Up to a rotation, we can also ask that

$$\mathcal{H}^{n-1}\left(\{x \in \mathcal{F}E : \ \nu_E(x) = \pm e_k, \ k = 1, ..., n\}\right) = 0. \tag{3.32}$$

Now we can apply Theorem 3.15 to E, using hyperplanes parallel to the coordinate axes. The resulting set is constructed by reflectinf n times a certain subset of E with respect to the coordinate directions, and therefore it still satisfies the condition (3.32), while being contained in $[-L(n), L(n)]^n$.

Summarizing, we can assume that $|E| = |B|$, $\delta(E) \leq \delta(n)$, $E \subset [-L(n), L(n)]^n$, E is indecomposable, n-symmetric with respect to the coordinate axes and (3.32) holds true. Under these assumptions we are going to show that

$$|E \Delta B| \leq C(n)\sqrt{\delta(E)}. \tag{3.33}$$

The result then follows from $|E| = |B| = C(n)$ and the fact that $A(E) \leq |E\Delta B| \,/\, |E|$.

We consider the stripes $S_1 = \left\{x : \ |x_1| < \frac{\sqrt{2}}{2}\right\}$ and $S_2 = \left\{x : \ |x_2| < \frac{\sqrt{2}}{2}\right\}$. Now, since $B \subset S_1 \cup S_2$, and $(B \setminus E) \cap S_1^c \subset (B \setminus E) \cap S_2$, we get

$$|(B \setminus E) \cap S_1| + |(B \setminus E) \cap S_2| \geq |B \setminus E|\,,$$

which implies that either $|(B \setminus E) \cap S_1| \geq |B \setminus E| \,/2$ or $|(B \setminus E) \cap S_2| \geq |B \setminus E| \,/2$. We can assume the first case holds, up to a rotation. Notice that such a rotation only changes the vectors e_1 and e_2 by a permutation and a sign, and therefore (3.32) is still in force. Since $|E| = |B|$, we have $|E \setminus B| = |B \setminus E|$ and the following bound holds

$$|E\Delta B| = |E \setminus B| + |B \setminus E| = 2\,|B \setminus E| \leq 4\,|(B \setminus E) \cap S_1|\,. \tag{3.34}$$

Now let E^* be the Schwarz symmetral of E with respect to the x_1-axis. Apart from being axially symmetric, E^* is also symmetric to the coordinate hyperplanes, since E is a well. Additionally, the Schwarz inequality (Theorem 3.9) implies that $P(E^*) \leq P(E)$, which yields

$$\delta(E^*) \leq \delta(E). \tag{3.35}$$

We now use E^* as a comparison object between E and B,

$$|(B \setminus E) \cap S_1| \leq |(E^* \setminus E) \cap S_1| + |(B \setminus E^*) \cap S_1|\,, \tag{3.36}$$

and our aim is to show that

$$|(E^* \Delta E) \cap S_1| \leq C(n)\sqrt{\delta(E)}, \tag{3.37}$$

$$|(B \Delta E^*) \cap S_1| \leq C(n)\sqrt{\delta(E^*)}. \tag{3.38}$$

Combining these two estimates with (3.36), (3.34), and (3.35) yields the result. Therefore we divide the rest of the proof into three steps. In the first one we will achieve a lower bound for the function $v(t) = \mathcal{H}^{n-1}\left(E \cap \{x_1 = t\}\right)$, which we will need for the following steps. In the second one we will prove (3.38) with an argument inspired by Gromov's proof of the isoperimetric inequality, which we will discuss below in section 4.2 in greater detail and generality. Finally, we will use an inductive argument to prove (3.37), thus completing the proof.

Step 1. A lower bound on v. Let $w(t) = \mathcal{H}^{n-1}(B \cap \{x_1 = t\})$, and let $c_0 = c_0(n) = w(\lambda)$ for some fixed $\lambda \in (\sqrt{2}/2, 1)$. We claim that, provided $\delta(n)$ is small enough, we have

$$\inf\{v(t) : \ t \in I\} \geq \frac{c_0}{3}, \quad I = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right). \tag{3.39}$$

Assume for contradiction that $t_0 \in I$ is such that $v(t_0) < c_0/3$, and consider (t_1, t_2) to be the smallest interval contained in $(-\lambda, \lambda)$, in which t_0 lies and such that

$$v(t) \leq \frac{c_0}{2}, \quad \forall t \in (t_1, t_2).$$

Since $w(t) \geq c_0$ for $t \in (-\lambda, \lambda)$ and $t_1 < t_0 < t_2$,

$$\frac{c_0}{2}(t_2 - t_1) \leq \int_{\mathbb{R}} |w(t) - v(t)| \, \mathrm{d}t = |E^* \Delta B| \leq 3 \, |B| \, A(E^*),$$

where in the last inequality we used Lemma 3.12 and the fact that E^* is n-symmetric with the origin as the center of symmetry. Now using Theorem 3.6 and the fact that $\delta(E^*) \leq \delta(E) \leq \delta(n)$, we have

$$t_2 - t_1 \leq C(n)A(E) \leq \omega(\delta(n)), \tag{3.40}$$

for some continuous function $\omega : [0, \infty) \longrightarrow [0, \infty)$, independent of E^* and such that $\omega(t) \to 0$ as $t \to 0^+$. Notice additionally that by the continuity of v (due to (3.32)), we have

$$v(t_1) = v(t_2) = \frac{c_0}{2}.$$

Now let F be the axially symmetric set such that $\mathcal{H}^{n-1}(F \cap \{x_1 = t\}) = v(t)$ outside (t_1, t_2), and that takes constant value $c_0/2$ on (t_1, t_2). Also set $S = (t_1, t_2) \times \mathbb{R}^{n-1}$. Then

$$P(E^*) = P(F) - P(F|S) + P(E^*|S).$$

We seek to control the three terms on the right-hand side and find a contradiction with the small isoperimetric deficit assumption for E^*.

The third term can be bounded from below by arguing as in the proof of (3.19),

$$\begin{aligned} P(E^*|S) &\geq \int_{t_1}^{t_2} \sqrt{\mathcal{H}^{n-1}(\mathcal{F}E_t^*)^2 + v'(t)} \, \mathrm{d}t \geq \int_{t_1}^{t_2} |v'| \\ &\geq |v(t_1) - v(t_0)| + |v(t_0) - v(t_1)| = \frac{c_0}{3}. \end{aligned}$$

The second term clearly has $P(F|S) = (c_0/2)\,|t_1 - t_2| \leq c(n)\omega(\delta(n))$. For the third term we use the isoperimetric inequality and the fact that $E^* \subset F$,

$$P(F) \geq n \, |B|^{1/n} \, |F|^{1/n'} \geq n \, |B|^{1/n} \, |E^*| \, .$$

Finally, combining these three facts we find

$$P(E^*) \geq n \, |B|^{1/n} \, |E^*|^{1/n'} + \frac{c_0}{3} - c(n)\omega(\delta(n)),$$

which, when expressed in terms of the isoperimetric deficit yields

$$\delta(E^*) \geq \frac{c_0}{3C(n)} - \frac{\omega(\delta(n))}{C(n)}.$$

Finally, since $\delta(n) \geq \delta(E^*)$, taking $\delta(n)$ small enough we find a contradiction.

Step 2. Estimate for $|E^* \Delta B \cap S_1|$. The proof for this step is based on a parametrization of B in terms of E through the function τ defined by

$$|E^* \cap \{x_1 < t\}| = |E \cap \{x_1 < t\}| = |B \cap \{x_1 < \tau(t)\}|. \tag{3.41}$$

Since E is indecomposable and bounded, and v is continuous, then $\{t \in \mathbb{R} : \ v(t) > 0\}$ is an interval. By n-symmetry (in particular with respect to the hyperplane perpendicular to $\{x_1 = 0\}$), $\{v > 0\} = (-M, M)$. Thanks to the lower bound (3.39) we can assume $M > \sqrt{2}/2$ (i.e. $I \subset (-M, M)$). It turns out that (3.41) defines the function $\tau : (-M, M) \longrightarrow (-1, 1)$, and by construction,

$$\int_{-\infty}^{t} v(s)\,\mathrm{d}s = \int_{-\infty}^{\tau(t)} w(s)\,\mathrm{d}s. \tag{3.42}$$

Then by considering difference quotients it can easily be proved that τ is differentiable, with

$$\tau'(t) = \frac{v(t)}{w(\tau(t))}, \quad \forall t \in (-M, M), \tag{3.43}$$

making $\tau \in C^1((-M, M))$ and $\tau' > 0$. Furthermore, as v is *absolutely continuous* (Lemma 3.10) and w and τ are a.e. differentiable, we also find that a.e. $t \in (-M, M)$ there exists $\tau''(t)$, and a simple computation shows

$$\tau''(t) = \frac{v'(t)w(\tau(t))^2 - w'(\tau(t))v(t)^2}{(\tau(t))^3}.$$

Since $I \subset (-M, M)$ and w is bounded on $[-1, 1]$ and uniformly Lipschitz on I, we find

$$\begin{aligned}
|(E^* \Delta B) \cap S_1| &= \int_I |v(t) - w(t)|\,\mathrm{d}t = \int_I |w(\tau(t))\tau'(t) - w(t)|\,\mathrm{d}t \\
&= \int_I |w(\tau(t))\tau'(t) - w(\tau(t)) + w(\tau(t)) - w(t)|\,\mathrm{d}t \\
&\leq \int_I w(\tau(t))\,|\tau'(t) - 1|\,\mathrm{d}t + \int_I |w(\tau(t)) - w(t)|\,\mathrm{d}t \\
&\leq C(n) \int_I |\tau'(t) - 1| + |\tau(t) - t|\,\mathrm{d}t \leq C(n) \int_I |\tau' - 1|\,,
\end{aligned} \tag{3.44}$$

where in the last step we used the fact that $\tau(0) = 0$ to deduce $|\tau(t) - t| \leq \int_I |\tau' - 1|$ for $t \in I$. In order to conclude the proof of *Step 2*, we need to estimate $||\tau' - 1||_{L^1(I)}$ by $\sqrt{\delta(E^*)}$. To this end, let us consider the deformation

$$\begin{array}{rccc}
T : (-M, M) \times \mathbb{R}^{n-1} & \longrightarrow & \mathbb{R}^n \\
x & \longmapsto & \tau(x_1)e_1 + \sum_{i=2}^n \left(\frac{w(\tau(x_1))}{v(x_1)}\right)^{1/(n-1)} x_i e_i.
\end{array}$$

This is, in fact, a Knothe-like construction (see Figure 7), where we require τ to fulfill (3.41) (that is, to send the section $E^*_{x_1}$ into $B_{\tau(x_1)}$) and then, given the axial symmetry of E^*, we can choose the map $T = (\tau(x_1), M_2(x_1, x_2), ..., M_n(x_1, x_n))$ so that if we let $E^{*j}_{x_1}$ be the intersection of $E^*_{x_1}$ with the x_j-axis, then

$$\frac{\mathcal{H}^1\left(E^{*j}_{x_1} \cap \{x_j < t\}\right)}{\mathcal{H}^1\left(E^{*j}_{x_1}\right)} = \frac{\mathcal{H}^1\left(B^j_{\tau(x_1)} \cap \{x_j < M_j(x_1, t)\}\right)}{\mathcal{H}^1\left(B^j_{\tau(x_1)}\right)}, \quad t \in \mathbb{R}.$$

We are using T to send the 1-dimensional sections of $E^*_{x_1}$ into those of $B_{\tau(x_1)}$, and T ends up being a transport map between E^* and B. By (3.43), and since $1/(n-1) = n'/n$, we can also write

$$T(x) = \tau(x_1)e_1 + \sum_{i=2}^{n} \frac{1}{\tau'(x_1)^{n'/n}} x_i e_i, \quad \forall x \in (-M, M) \times \mathbb{R}^{n-1}.$$

We can differentiate T at a.e. $x \in (-M, M) \times \mathbb{R}^{n-1}$, obtaining

$$\nabla T = \tau'(x_1)e_1 \otimes e_1 + \sum_{i=2}^{n} \frac{1}{\tau'(x_1)^{n'/n}} e_i \otimes e_i + \left(\sum \frac{\partial \frac{1}{\tau'(x_1)^{n'/n}}}{\partial x_1} \right) \otimes e_1,$$

and ∇T has at most two distinct eigenvalues: $\tau'(x_1)$ in the direction of x_1, and $\frac{1}{\tau'(x_1)^{n'/n}}$ on e_i, for $i \neq 1$. Therefore we find

$$\operatorname{div} T(x) = \tau'(x_1) + (n-1)\frac{1}{\tau'(x_1)^{n'/n}}.$$

A direct application of Young's inequality for products yields

$$\frac{\operatorname{div} T(x)}{n} = \frac{1}{n}(\tau'(x_1)^{1/n})^n + \frac{1}{n'}\left(\frac{1}{\tau'(x_1)^{1/n}}\right)^{n'} \geq 1.$$

Observe that $T(x) \in B$, and apply the Divergence Theorem to obtain

$$P(E^*) = \mathcal{H}^{n-1}(\mathcal{F}E^*) \geq \int_{\mathcal{F}E^*} T \cdot \nu_E \, \mathrm{d}\mathcal{H}^{n-1} = \int_{E^*} \operatorname{div} T \geq n \, |E^*| = n \, |B|^{1/n} \, |E^*|^{1/n'} . \tag{3.45}$$

For a rigorous justification of this step, see [Ma2, Appendix]. At this point we remark that we have followed the ideas from Gromov's proof of the isoperimetric inequality. If we had $\operatorname{div} T = n$, then using $\det \nabla T = 1$ (which follows from a simple computation) and the Divergence Theorem we would arrive to the equality case: E^* would be optimal. Since we want to *quantitatively* estimate how far this set is from being optimal, we estimate the distance between $\operatorname{div} T$ and n,

$$|B| \, \delta(E^*) \geq \int_{E^*} \left(\frac{\operatorname{div} T}{n} - 1 \right) \geq \int_{-M}^{M} \left(\frac{1}{n}\tau'(t) + \frac{n-1}{n} \frac{1}{\tau'(t)^{n'/n}} - 1 \right) v(t) \, \mathrm{d}t.$$

Setting $\sigma = (\tau')^{n'/n}$, we notice that

$$C(n)\delta(E^*) \geq \int_{-M}^{M} \frac{v}{\sigma} \left(\frac{\sigma^n}{n} + \frac{1}{n'} - \sigma \right) \mathrm{d}t \geq c(n) \int_{-M}^{M} \frac{v}{\sigma} (\sigma - 1)^2 \, \mathrm{d}t.$$

By the Hölder inequality,

$$\int_{-M}^{M} v \, |\sigma - 1| \leq C(n) \sqrt{\int_{-M}^{M} \frac{v}{\sigma}(\sigma-1)^2} \sqrt{\int_{-M}^{M} v\sigma} \leq C(n)\sqrt{\delta(E^*)}, \tag{3.46}$$

where we have used the previous estimate to introduce $\sqrt{\delta(E^*)}$, and the constant $C(n)$ appears since

$$\int_{-M}^{M} v\sigma \leq \left(\int_{-M}^{M} \tau' \right)^{n'/n} \left(\int_{-M}^{M} v^{\frac{n-1}{n-2}} \right)^{\frac{n-2}{n-1}} \leq C(n),$$

as $\int_{-M}^{M} \tau' = 2$ and $||v||_{L^1(\mathbb{R})} = |E|$, $||v||_{L^\infty(\mathbb{R})} \leq (2L(n))^{n-1}$. Notice how Theorem 3.5 allows us to control this last norm in terms of n, which would not be possible by directly applying Theorem 1.11.

Now, turning back to (3.46) and using the lower bound on v over I provided by *Step 1*, namely (3.39), we deduce

$$C(n)\sqrt{\delta(E^*)} \geq \int_I |\sigma - 1| = \int_I \left|(\tau')^{n'/n} - 1\right| \geq \int_I |\tau' - 1|,$$

where in the last step we have assumed boundedness of $\tau'(t)$ over a.e. $t \in I$. Therefore, as soon as we show that $||\tau'||_{L^\infty(I)} \leq C(n)$ we can use this, together with the previous estimate (3.44) to conclude the proof of *Step 2*.

By (3.43) and thanks to v and w being bounded, it is enough to show that for some constant $\lambda(n) \in (0,1)$ we have $\tau(t) \in [-\lambda(n), \lambda(n)]$ for every $t \in I$. Since $\tau' > 0$, τ is increasing, and therefore we only need to show that $\tau(\sqrt{2}/2) < c(n)$. Indeed, on the one hand,

$$\left|E^* \cap \left\{x_1 > \sqrt{2}/2\right\}\right| = \left|B \cap \left\{x_1 > \sqrt{2}/2\right\}\right| - |E \Delta B| \geq c_1(n) - 3\,|B|\,A(E^*) \geq c_2(n).$$

On the other hand,

$$\left|E^* \cap \left\{x_1 > \sqrt{2}/2\right\}\right| = \left|B \cap \left\{x_1 > \tau(\sqrt{2}/2)\right\}\right| \leq \left|B \cap \mathbb{R}^{n-1}\right| \left[1 - \tau(\sqrt{2}/2)\right].$$

Therefore $1 - \tau(\sqrt{2}/2) \geq c(n)$, which means that $\tau(\sqrt{2}/2) \leq 1 - c(n)$, and the proof of *Step 2* is complete.

Step 3. We show that if the theorem is valid in dimension $n-1$, then (3.37) holds true, meaning that it holds in dimension n. Let us recall from the proof of (3.9) that

$$P(E) \geq \int_{\mathbb{R}} \sqrt{p(t)^2 + v'(t)^2}\,\mathrm{d}t, \quad P(E^*) = \int_{\mathbb{R}} \sqrt{p_*(t)^2 + v'(t)^2}\,\mathrm{d}t, \quad p(t) \geq p_*(t).$$

Note that here we have set for brevity

$$p(t) = \mathcal{H}^{n-2}\left(\mathcal{F}E_t\right), \quad p_*(t) = \mathcal{H}^{n-2}\left(\mathcal{F}E_t^*\right).$$

As we are assuming (3.7) in dimension $n-1$, we know that for a.e. $t \in (-M, M)$, the Fraenkel asymmetry of E_t in $\mathbb{R}^{n-1}$, which we will denote by $A_{n-1}(E_t)$, is bounded,

$$A_{n-1}(E_t) \leq \sqrt{\frac{p(t)}{p_*(t)} - 1}.$$

As E is symmetric with respect to the coordinate axes, E_t is $(n-1)$-symmetric with respect to the coordinate axes as well. In particular, up to a multiplicative factor 3, we know that $A_{n-1}(E_t)$ is equivalent to the distance of E_t from E_t^*, the $(n-1)$-dimensional ball with the same $\mathcal{H}^{n-1}$-dimensional measure as E_t, centered at te_1, and oriented by e_1. We thus come to the estimate

$$\frac{\mathcal{H}^{n-1}\left(E_t \Delta E_t^*\right)}{v(t)} \leq C(n)\sqrt{\frac{p(t) - p_*(t)}{p_*(t)}}.$$

We are now in position to prove (3.37). On the one hand,

$$|(E^*\Delta E)\cap S_1| = \int_I \mathcal{H}^{n-1}(E_t\Delta E_t^*)\,\mathrm{d}t \le C(n)\int_I \sqrt{\frac{p(t)-p_*(t)}{p_*(t)}}\,\mathrm{d}t, \tag{3.47}$$

since $||v||_{L^\infty(\mathbb{R})} \le (2L(n))^{n-1}$. On the other hand,

$$\delta(E) = \frac{P(E)}{P(B)} - 1 = \frac{P(E)-P(E^*)}{P(B)} + \delta(E^*),$$

so that

$$\begin{aligned} P(B)\delta(E) \ge P(E) - P(E^*) &\ge \int_{\mathbb{R}} \sqrt{p^2+(v')^2} - \sqrt{p_*^2+(v')^2} \\ &= \int_{\mathbb{R}} \frac{p^2-p_*^2}{\sqrt{p^2+(v')^2}-\sqrt{p_*^2+(v')^2}}. \end{aligned}$$

Now, using the Hölder inequality together with the previous estimations we find

$$\begin{aligned} \int_{\mathbb{R}} \sqrt{p^2-p_*^2} &\le \sqrt{\int_{\mathbb{R}} \frac{p^2-p_*^2}{\sqrt{p^2+(v')^2}-\sqrt{p_*^2+(v')^2}}}\sqrt{\int_{\mathbb{R}} \sqrt{p^2+(v')^2}-\sqrt{p_*^2+(v')^2}} \\ &\le \sqrt{P(B)\delta(E)}\sqrt{P(E)+P(E^*)}. \end{aligned}$$

As $P(E)+P(E^*) \le 2P(E) \le 2P(B)(1+\delta(E)) \le C(n)$, we have

$$C(n)\sqrt{\delta(E)} \ge \int_{\mathbb{R}} \sqrt{p^2-p_*^2} \ge \int_{\mathbb{R}} p_*\sqrt{\frac{p}{p_*}-1}.$$

Thanks to (3.47), we find

$$C(n)\sqrt{\delta(E)} \ge \inf\{p_*(t) :\ t\in I\}\,|(E\Delta E^*)\cap S_1| \ge c(n)\,|(E^*\Delta E)\cap S_1|\,,$$

where we have used (3.39) and the fact that $p_*(t)$ is a power of $v(t)$. Hence we have proven the inductive step.

We now remark that since the theorem is valid in dimension one, the statement we just proved implies the desired result via an induction argument, and Theorem 3.3 is proved. □

4 Anisotropic isoperimetric inequality

As remarked in the introduction, the notion of anisotropic perimeter comes as a generalization of the classical perimeter (discussed in section 1.1). In dimension $n \geq 2$, we consider an open, bounded, convex set $K \subset \mathbb{R}^n$ containing the origin, and we define, for $x \in \mathbb{R}^n$,

$$||x|| = \inf\left\{\lambda > 0 : \ \frac{x}{\lambda} \in K\right\},$$

which characterizes K as the set $\{x \in \mathbb{R} : \ ||x|| < 1\}$. If K is the unit ball B, then $||x|| = |x|$, but in general this quantity is not a norm, since K may fail to allow for $||x|| = ||-x||$ to hold. In any case, we can define the following function through the scalar product,

$$||y||_* := \sup\{x \cdot y : \ ||x|| \leq 1\}, \quad y \in \mathbb{R}^n. \tag{4.1}$$

Apart from giving us the Cauchy-Schwarz type inequality

$$x \cdot y \leq ||x||\,||y||_*, \quad x, y \in \mathbb{R}^n, \tag{4.2}$$

we can think of it acting on $\mathbb{S}^{n-1}$ as a weighting function through which it is possible to define a generalized notion of the classical perimeter. Similarly to how $P(E) = |\mu_E|(\mathbb{R}^n)$ and $|\mu_E| = \mathcal{H}^{n-1}|_{\mathcal{F}E}$, we can define $P_K(E)$ by weighting against $||\nu_E||_*$ instead of $|\nu_E|$.

Definition 4.1. If E is a set of finite perimeter in $\mathbb{R}^n$, we define the *anisotropic perimeter* of E as

$$P_K(E) := \int_{\mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1}. \tag{4.3}$$

In particular, when the set K is the Euclidean ball B we recover the classical Euclidean perimeter. It is therefore reasonable to think that an isoperimetric-type inequality holds in the anisotropic case, and indeed we have the following result.

Theorem 4.2 (Anisotropic isoperimetric inequality)**.** *If E is a set of finite perimeter with $|E| < \infty$, we have*

$$n\,|K|^{1/n}\,|E|^{1/n'} \leq P_K(E). \tag{4.4}$$

In attempting to prove Theorem 4.2, we find that the symmetrization techniques employed in the Euclidean case no longer succeed now, since the set K does not generally enjoy any kind of symmetry. Although anisotropic symmetrization techniques can be used to tackle the problem [VSc], one straightforward way to get around this is to rely on the theory of (optimal) mass transportation. In particular, Gromov's proof of the isoperimetric inequality [Gr, Appendix I] can be adapted with slight variations to reach a proof of Theorem 4.2. Before presenting it, we need to introduce some preliminary concepts and results concerning functions of bounded variation and the anisotropic perimeter.

4.1 Functions of bounded variation and anisotropic perimeter

We can endow the space of $n \times n$ tensors $\mathbb{R}^{n\times n}$ with the metric

$$|A| = \sqrt{\operatorname{trace}(A^t A)}, \quad A \in \mathbb{R}^{n\times n}.$$

In particular, if $T \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and DT is its $\mathbb{R}^{n\times n}$-valued distributional derivative, then we denote by $|DT|$ the total variation of DT with respect to this metric.

Definition 4.3. Given $n \geq 2$, we define $BV(\mathbb{R}^n;\mathbb{R}^n)$ to be the space of $L^1(\mathbb{R}^n;\mathbb{R}^n)$ vector fields T such that $|DT|(\mathbb{R}^n) < \infty$.

Using the Lebesgue-Besicovitch Differentiation Theorem [Ma1, Theorem 5.8], we denote by ∇T the density of DT with respect to the Lebesgue measure, and by $D_s T$ the corresponding singular part, meaning that we decompose

$$DT = \nabla T \, \mathrm{d}x + D_s T. \tag{4.5}$$

We denote by $\operatorname{Div} T$ the distributional divergence of T, and notice that if DT takes values in $\mathbb{R}^{n\times n}$ and is symmetric and positive definite, then, by a diagonalization argument, $\operatorname{Div} T$ is a non-negative Radon measure on $\mathbb{R}^n$, which is bounded from above and below by the total variation of T, i.e., for every Borel set C in $\mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} \operatorname{Div} T(C) \leq |DT|(C) \leq \operatorname{Div} T(C), \tag{4.6}$$

since

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \lambda_i \leq \left(\sum_{i=1}^n \lambda_i^2 \right)^{1/2} \leq \sum_{i=1}^n \lambda_i, \quad \lambda_i \geq 0.$$

Moreover, setting $\operatorname{div} T(x) = \operatorname{trace}(\nabla T(x))$, and taking the trace in (4.5),

$$\operatorname{Div} T = \operatorname{div} T \, \mathrm{d}x + (\operatorname{Div} T)_s, \quad (\operatorname{Div} T)_s = \operatorname{trace}(D_s T) \geq |D_s T|. \tag{4.7}$$

As a consequence, $\operatorname{Div} T - \operatorname{div} T \, \mathrm{d}x$ is a non-negative Radon measure, which means that for any measurable set E,

$$\int_E \operatorname{div} T \, \mathrm{d}x \leq \operatorname{Div} T(E). \tag{4.8}$$

Whenever $T \in BV(\mathbb{R}^n;\mathbb{R}^n)$ and E is a set of finite perimeter, we can find a representation of T along $\mathcal{F}E$ called the *inner trace of T on E* [AFP, Theorem 3.77]. That is, for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E$ there exists a vector $\operatorname{tr}_E(T)(x) \in \mathbb{R}^n$ such that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x,r)\cap\{y:\ (y-x)\cdot\nu_E(x)<0\}} |T(y) - \operatorname{tr}_E(T)(x)| \, \mathrm{d}y = 0. \tag{4.9}$$

What is more, as a consequence of [AFP, Example 3.97] applied to T and $\mathbb{1}_E$, the Divergence Theorem holds in the form

$$\operatorname{Div} T(E^{(1)}) = \int_{\mathcal{F}E} \operatorname{tr}_E(T) \cdot \nu_E \, \mathrm{d}\mathcal{H}^{n-1}, \tag{4.10}$$

whenever $T \in BV(\mathbb{R}^n;\mathbb{R}^n)$ and E is a set of finite perimeter.

4.1.1 Anisotropic perimeter

Similarly to how the total variation of a $\mathbb{R}^n$-valued Borel measure is defined, we have its *anisotropic total variation*.

Definition 4.4. If μ is a $\mathbb{R}^n$-valued Borel measure, its anisotropic total variation $||\mu||_*$ is the nonnegative Borel measure defined on the Borel set E as

$$||\mu||_*(E) = \sup\left\{\sum_{h\in\mathbb{N}} ||\mu(E_h)||_* : \ E_k \cap E_h = \emptyset, \ \bigcup_{h\in\mathbb{N}} E_h \subset E.\right\}$$

If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define its anisotropic total variation as

$$TV_K(f) := \sup\left\{\int_{\mathbb{R}^n} \operatorname{div} T(x) f(x)\,\mathrm{d}x : \ T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n), \ ||T|| \le 1\right\}. \tag{4.11}$$

In particular, if A is an open set in $\mathbb{R}^n$, then the definition agrees with

$$||\mu||_*(A) = \sup\left\{\int_{\mathbb{R}^n} T\cdot \mathrm{d}\mu : \ T \in C^1(A,\mathbb{R}^n), \ ||T|| \le 1\right\}.$$

If $f \in BV(\mathbb{R}^n)$ then $TV_K(f) = ||-Df||_*(\mathbb{R}^n)$. When $K = B$, then $TV_K(f) = |Df|(\mathbb{R}^n)$, and since K is bounded, open and contains the origin, we can find $r, R > 0$ for which $B_r \subset K \subset B_R$. By a rescaling argument, $TV_K(f) < \infty$ if and only if $TV_B(f) = |Df|(\mathbb{R}^n) < \infty$.

We now focus on a discussion similar to the one in section 1.1 and remark the various similarities that both the anisotropic and the classical notions of perimeter share in common. In particular, the same idea of approximating $P(E)$ by regularizing $\mathbb{1}_E$ can be applied to the anisotropic case, and an identical approximation result by bounded, smooth sets holds in the anisotropic version.

If E is a set of finite perimeter, then recovering the notation introduced in section 1.1,

$$P_K(E) = ||\mu_E||_*(\mathbb{R}^n), \quad \text{and} \quad ||\mu_E||_* = ||\nu_E||_* \mathcal{H}^{n-1}\Big|_{\mathcal{F}E}.$$

Additionally, if $\mathbb{1}_E$ is its characteristic function, then $TV_K(\mathbb{1}_E) = P_K(E)$, and if $f \in C^1_c(\mathbb{R}^n)$,

$$TV_K(f) = \int_{\mathbb{R}^n} ||-\nabla f(x)||_* \,\mathrm{d}x. \tag{4.12}$$

The reason why $-\nabla f(x)$ appears in (4.12) is that it is parallel to the outer normal direction to $\{f > f(x)\}$. In this fashion,

$$P_K(E) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} ||-\nabla \mathbb{1}_{E,\varepsilon}||_*,$$

just like in the Euclidean version of the perimeter. Analogously, we define the *anisotropic perimeter of E relative to A* by

$$P_K(E|A) = ||-D\mathbb{1}_E||_*(A) = ||\mu_E||_*(A) = \sup\left\{\int_E \operatorname{div} T(x)\,\mathrm{d}x : \ T \in C^1_c(A;K)\right\}.$$

This functional enjoys similar properties to those of $P(E|A)$, to name a few it coincides with $P_K(E)$ when $A = \mathbb{R}^n$, it is lower semicontinuous with respect to the local convergence of sets, and in general,

$$P_K(E|A) = \int_{A \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1}.$$

Relative perimeters appear in the Fleming-Rishel Coarea Formula, the anisotropic version of Theorem 1.12.

Theorem 4.5 (Anisotropic Coarea Formula)**.** *If $A \subset \mathbb{R}^n$ is an open set and $f \in C^1(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$, then*

$$\int_A ||-\nabla f(x)||_* \, \mathrm{d}x = \int_{\mathbb{R}} P_K(\{f > t\} \, |A) \, \mathrm{d}t. \tag{4.13}$$

More generally, if additionally $\psi : \mathbb{R}^n \longrightarrow [0, \infty]$ is a Borel function, then

$$\int_{\mathbb{R}^n} \psi \, \mathrm{d}||-Df||_* = \int_{\mathbb{R}} \mathrm{d}t \int_{\mathcal{F}\{f>t\}} \psi \left|\left|\nu_{\{f>t\}}\right|\right|_* \mathrm{d}\mathcal{H}^{n-1}. \tag{4.14}$$

As in the Euclidean version, this instance of the Coarea Formula allows for a similar approximation result to Theorem 1.11.

Theorem 4.6. *If E is a set of finite perimeter in $\mathbb{R}^n$, then there exists a sequence $\{E_h\}_{h=1}^{\infty}$ of bounded, open sets with smooth or polyhedral boundary such that $E_h \to E$ and $P_K(E_h) \to P_K(E)$.*

4.2 Proof of the anisotropic isoperimetric inequality

We now come to the proof of Theorem 4.2.

Proof of Theorem 4.2. We may assume that E is of finite perimeter with $|E| = |K|$. By the Brenier-McCann Theorem [Vi, Theorem 2.32], suitably modified to take into account the fact that K is bounded (see p. 19 in [MV, Section 1.2]), there exists a convex, continuous function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\nabla\varphi(x) \in K$ for a.e. $x \in \mathbb{R}^n$ and $\nabla\varphi_{\#}\left(\mathbb{1}_E(x) \, \mathrm{d}x\right) = \mathbb{1}_K(y) \, \mathrm{d}y$. Let us set $T = \nabla\varphi$, so that

$$\int_K h(y) \, \mathrm{d}y = \int_E h\left(T(x)\right) \mathrm{d}x \tag{4.15}$$

for every Borel function $h : \mathbb{R}^n \longrightarrow [0, \infty]$. Since T is the gradient of a convex function, its distributional derivative DT takes values in the set of symmetric and positive semidefinite $n \times n$ tensors. Therefore, $T \in BV(\mathbb{R}^n; K)$ by [AA, Proposition 5.1].

Starting from (4.15), and since ∇T is locally invertible a.e. on E (see [McC]), we can use h to localize around points of E so that, changing variables $y = T(x)$,

$$\int h(T(x)) \left(\det \nabla T(x) - \mathbb{1}_E(x)\right) \mathrm{d}x = 0,$$

We deduce that

$$\det \nabla T(x) = 1, \quad \text{a.e. } x \in E.$$

Since $\nabla T(x)$ is a positive semidefinite symmetric tensor for a.e. $x \in \mathbb{R}^n$, we can find measurable functions $\lambda_k : \mathbb{R}^n \longrightarrow [0, \infty)$ and $e_k : \mathbb{R}^n \longrightarrow \mathbb{S}^{n-1}$, for $k = 1, ..., n$, such that

$$0 \leq \lambda_k \leq \lambda_{k+1}, \quad e_i \cdot e_j = \delta_{i,j}, \quad \nabla T = \sum_{k=1}^{n} \lambda_k e_k \otimes e_k,$$

with $\lambda_k(x) > 0$ for a.e. $x \in E$. The arithmetic-geometric mean inequality implies that, for a.e. $x \in E$,

$$n = n \left(\det \nabla T(x)\right)^{1/n} = n \left(\prod_{k=1}^{n} \lambda_k(x) \right)^{1/n} \leq \sum_{k=1}^{n} \lambda_k(x) = \operatorname{div} T(x). \tag{4.16}$$

Now, using (4.16), (4.8) and the Divergence Theorem (1.8), we find

$$n \left|K\right|^{1/n} \left|E\right|^{1/n} = n \left|E\right| = \int_E n \left(\det \nabla T(x)\right)^{1/n} \mathrm{d}x \tag{4.17}$$

$$\leq \int_E \operatorname{div} T(x) \, \mathrm{d}x = \int_{E^{(1)}} \operatorname{div} T(x) \, \mathrm{d}x \tag{4.18}$$

$$\leq \operatorname{Div} T \left(E^{(1)}\right) = \int_{\mathcal{F}E} \operatorname{tr}_E(T) \cdot \nu_E \, \mathrm{d}\mathcal{H}^{n-1}. \tag{4.19}$$

Since T takes values in K, we find by (4.9) that $||\operatorname{tr}_E(T)(x)|| \leq 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E$. Finally, using (4.2) we conclude that

$$n \left|K\right|^{1/n} \left|E\right|^{1/n'} \leq \int_{\mathcal{F}E} ||\operatorname{tr}_E(T)|| \, ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1} \leq \int_{\mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1} = P_K(E). \tag{4.20}$$

□

5 Stability of the anisotropic isoperimetric inequality

In the case of the anisotropic isoperimetric inequality, an identical stability result to Theorem 3.3 can be established. In a similar vein to how we introduced the isoperimetric deficit and the Fraenkel asymmetry before, we now define their respective anisotropic versions. Throughout this section we will again assume that $K \subset \mathbb{R}^n$ is an open, bounded, convex set containing the origin.

Definition 5.1. Let $0 < |E| < \infty$ be a measurable set in $\mathbb{R}^n$. We define its *anisotropic isoperimetric deficit* by

$$\delta(E) := \frac{P_K(E)}{n\,|K|^{1/n}\,|E|^{1/n'}} - 1.$$

The role of δ will again be that of measuring the failure of E being optimal in the anisotropic isoperimetric inequality. This time, however, optimality means that E is equivalent to K (up to a rescaling and a rigid movement), and not to a ball. A priori, this should not be clear, since we have not characterised the equality cases, unlike in the Euclidean case. In fact, it was first conjectured by Wulff that K was the unique minimizer for the anisotropic problem up to translations and scalings. This motivates the following alternative to the Fraenkel asymmetry.

Definition 5.2. Given a measurable set $0 < |E| < \infty$ in $\mathbb{R}^n$, we define its *asymmetry index* as

$$A(E) := \inf\left\{ \frac{|E\Delta(x_0 + rK)|}{|E|} :\ x_0 \in \mathbb{R}^n,\ r^n\,|K| = |E| \right\}.$$

As in the Euclidean case, the asymmetry index is invariant under the same operations that leave the deficit unchanged, and if K is the unit ball, we recover the same notions that we had in the Euclidean setting. In fact, Theorem 3.3 can be seen as a particular case of the following result.

Theorem 5.3. *Let E be a set of finite perimeter with $|E| < \infty$. Then,*

$$A(E) \leq C(n)\delta(E)^{1/2},$$

or, equivalently,

$$P_K(E) \geq n\,|K|^{1/n}\,|E|^{1/n'}\left[1 + \left(\frac{A(E)}{C(n)}\right)^2\right].$$

The first attempts at tackling this problem are due to [EFT], where a stability result

$$A(E) \leq C(n, K)\delta(E)^{\alpha(n)}$$

is proved for some constant that depends on n and K, and for the exponent

$$\alpha(n) = \begin{cases} \frac{9}{2}, & n = 2, \\ \frac{n(n+1)}{2}, & n \geq 3 \end{cases}$$

of $\delta(E)$, which is not optimal in the decay rate. The approach we will follow is that of [FiMP], where they prove the sharp inequality in Theorem 5.3. It makes use of mass transportation theory, and in particular it is based on a quantitative analysis of the Brenier map between E and K, which we used in the proof of Theorem 4.2. We will split the proof of Theorem 5.3 into several lemmas, and throughout this part we will often refer to the constants m_K and M_K, defined by

$$m_K := \inf\left\{||\nu||_* : \ \nu \in \mathbb{S}^{n-1}\right\}, \quad M_K := \sup\left\{||\nu||_* : \ \nu \in \mathbb{S}^{n-1}\right\}. \tag{5.1}$$

These satisfy, for every $x \in \mathbb{R}^n$,

$$\frac{|x|}{M_K} \leq ||x|| \leq \frac{|x|}{m_K}. \tag{5.2}$$

Before moving onto the results, we will dedicate the rest of this section to explain, or at least give some intuition on how we will prove Theorem 5.3.

Gromov's original proof of the isoperimetric inequality was based on the use of the Knothe map M between E and K. It the study of stability, however, the Knothe construction poses certain problems that can be avoided by using the Brenier map.

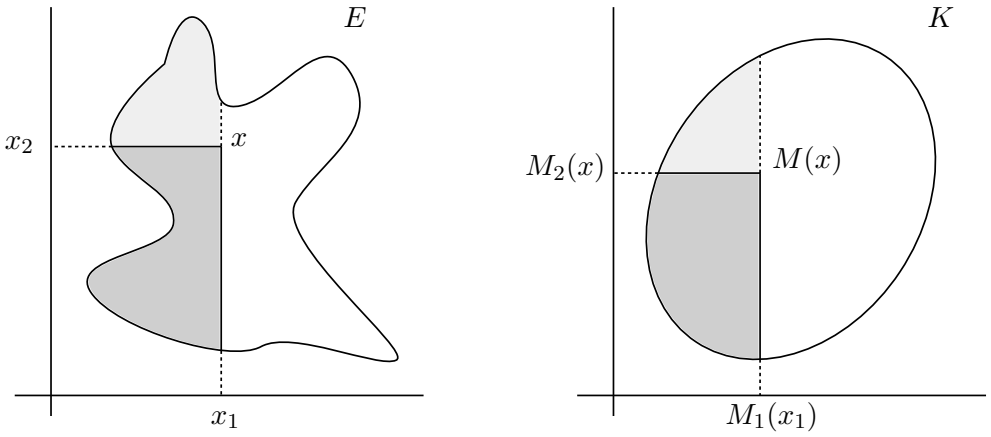


FIGURE 7: When constructing the Knothe map, the vertical section E_{x_1} is sent into $K_{M_1(x_1)}$, and $M_1(x_1)$ is chosen so that $|y \in E : y_1 < x_1| \,/\, |E|$ and $|z \in K : z_1 < M_1(x_1)| \,/\, |K|$ coincide. Then E_{x_1} gets displaced along $K_{M_1(x_1)}$ by choosing M_2 so that the $\mathcal{H}^1$-measure of $\{y \in E_{x_1} : y_2 < x_2\}$ relative to $\mathcal{H}^1(E_{x_1})$ and the $\mathcal{H}^1$-measure of $\{z \in K_{M_1(x_1)} : z_2 < M_2(x_1, x_2)\}$ relative to $\mathcal{H}^1(K_{M_1(x_1)})$ are equal.

Suppose E is optimal in (4.4), with the same barycenter as K, and let M be the Knothe map (see Figure 7) constructed using the canonical basis of $\mathbb{R}^n$, and T be the Brenier map. We can derive from Gromov's argument the conditions $n\,(\det \nabla M)^{1/n} = \operatorname{div} M$ and $n\,(\det T)^{1/n} = \operatorname{div} T$. From the first one we find that $\partial M_k/\partial x_k$ are all equal on E. Since $\det \nabla M = 1$, all $\partial M_k/\partial x_k$ are 1 on E. While this is not enough to deduce $\nabla M = \mathrm{id}$ since ∇M is upper triangular, it is still possible to prove that $E = K$ by means of the following argument.

Let $v(t) = \mathcal{H}^{n-1}(\{x \in E : \ x_1 = t\})$ and $w(t) = \mathcal{H}^{n-1}(\{x \in K : \ x_1 = t\})$. Given that $\partial M_1/\partial x_1 = 1$ on E, it follows that $u = v$. In particular, $\{u > 0\} = \{v > 0\}$ is an open

interval (α, β) and we find

$$|E \cap \{x \in \mathbb{R}^n : \ x_1 \in \mathbb{R} \setminus (\alpha, \beta)\}| = 0.$$

If we now fix a direction $\nu \in \mathbb{S}^{n-1}$, complete it into an orthonormal basis, apply the above argument to the corresponding Knothe map and repeat this process for every direction $\nu \in \mathbb{S}^{n-1}$, we find that $|E \setminus K| = 0$, and since $|E| = |K|$ this implies $E = K$. This approach, however, leads to a loss of optimality in the decay rate of the asymmetry index in terms of the isoperimetric deficit due to the usage of infinitely many Knothe maps.

The reason why this approach works in the proof of the sharp estimate in Theorem 3.3 is because E is reduced to an axially symmetric set and we only need to apply this argument once. Recall from that proof that we can then estimate the size of $E \setminus K$ by that of $(K \setminus E) \cap S_1$, and if E^* is the Schwarz symmetral of E with respect to the x_1-axis, then this last quantity can be bounded by $C(n)\sqrt{\delta(E)} + |(K \setminus E^*) \cap S_1|$. Using the above argument we conclude that $|(K \setminus E^*) \cap S_1| = 0$, which implies $A(E) = 0$. Of course, since in Theorem 3.3 we aim to prove stability, E is not optimal and these arguments are quantitative, achieving bounds in terms of $\delta(E)$.

The Brenier map allows us to avoid these difficulties. Since $\det \nabla T = 1$ on E and ∇T is symmetric, the condition $n\,(\det \nabla T)^{1/n} = \operatorname{div} T$ implies $\nabla T = \mathrm{id}$, and therefore $E = K$. In fact, using the proof of Theorem 4.2 we can argue quantitatively to obtain some initial bounds by $\delta(E)$ on the distance between T and the identity on E,

$$C(n)\,|K|\,\sqrt{\delta(E)} \geq \int_E |\nabla T - \mathrm{id}|.$$

If we could apply the Sobolev-Poincaré inequality on E, then we could bound by $C(n)\,|K|\,\sqrt{\delta(E)}$ the $L^{n'}$-norm of $T(x) - x$ over E, which could hopefully yield an estimate on $|E \Delta K|$. However, there is no reason to expect the necessary conditions for such an inequality to hold on E, since this set may not have a regular boundary or even be connected at all. It is here that the ideas from the previous part appear again: both the small deficit assumption and the reduction to better sets.

It turns out that, provided the set E is almost optimal, we can identify a maximal "critical subset" of E for the Sobolev-Poincaré inequality whose volume is controlled by $\delta(E)$. Removing this set means we can reduce E to another set where we can use the desired inequality. It would then remain to control $|E \Delta K|$ by the $L^{n'}$-norm of $T(x) - x$. It is not hard to find non-sharp estimates [FiMP, Section 1.6], but in order to prove Theorem 5.3 we need to use the optimality of T. This translates to a stronger reduction step: if E is almost optimal, up to removing a maximal critical subset, we can find a stronger trace inequality on E that implies

$$C(n, K)\sqrt{\delta(E)} \geq \int_{\mathcal{F}E} ||T(x) - x||\,||\nu_E(x)||_*\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

up to a translation of E. Since $||T(x)|| \leq 1$ on $\mathcal{F}E$ we have

$$|1 - ||x||| \leq |1 - ||T(x)||| + ||T(x) - x|| = (1 - ||T(x)||) + ||T(x) - x||\,,$$

and we can estimate $|1 - ||x|||$ by $||T(x) - x||$, which leads to the bound

$$C(n, K)\sqrt{\delta(E)} \geq \int_{\mathcal{F}E} |1 - ||x|||\,||\nu_E(x)||_*\,\mathrm{d}\mathcal{H}^{n-1}(x).$$

This last integral controls $|E \setminus K|$, and since $|E| = |K|$, we obtain control of $|E \Delta K|$. After bounding the constant $C(n, K)$ independently of K we achieve the proof of Theorem 5.3.

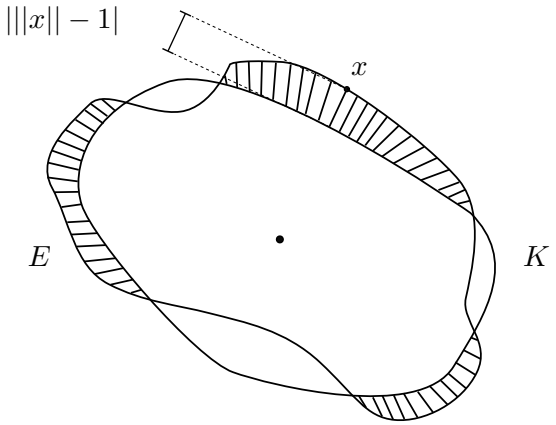


FIGURE 8: We can find a bound for $|E \setminus K|$ in terms of $\int_{\mathcal{F}E} |||x|| - 1| \, ||\nu_E(x)||_* \, \mathrm{d}\mathcal{H}^{n-1}(x)$.

5.1 Some initial bounds

From Gromov's proof of Theorem 4.2 we can deduce a few useful bounds on the distance of the Brenier map from a translation in terms of the isoperimetric deficit. In order to do that we first introduce the following lemma.

Lemma 5.4. *Let $0 < \lambda_1 \leq ... \leq \lambda_n$ be positive real numbers, and set*

$$\lambda_A = \frac{1}{n} \sum_{k=1}^{n} \lambda_k, \quad \lambda_G := \left(\prod_{k=1}^{n} \lambda_k \right)^{1/n}. \tag{5.3}$$

Then

$$7n^2 \left(\lambda_A - \lambda_G \right) \geq \frac{1}{\lambda_n} \sum_{k=1}^{n} \left(\lambda_k - \lambda_G \right)^2. \tag{5.4}$$

Proof. From the convexity of $t \mapsto 1/t$ over $t > 0$ we find

$$\log(s) \leq \log(t) + \frac{s-t}{t} - \frac{(s-t)^2}{2 \max\{s,t\}^2}, \quad s, t \in (0, \infty),$$

and using this we find that

$$\begin{aligned} \log(\lambda_G) = \frac{1}{n} \sum_{k=1}^{n} \log(\lambda_k) &\leq \frac{1}{n} \sum_{k=1}^{n} \left(\log(\lambda_A) + \frac{\lambda_k - \lambda_A}{\lambda_A} - \frac{(\lambda_k - \lambda_A)^2}{2\lambda_n^2} \right) \\ &= \log(\lambda_A) - \frac{1}{2n\lambda_n^2} \sum_{k=1}^{n} (\lambda_k - \lambda_A)^2 = \log(\lambda_A) - z, \end{aligned}$$

that is, $\lambda_G \leq \lambda_A e^{-z}$, where $z \in [0, 1/2]$. Using this, and since $1 - e^{-t} \geq 3t/4$ for every $t \in [0, 1/2]$, and $n\lambda_A \geq \lambda_n$, we have

$$\lambda_A - \lambda_G \geq \lambda_A(1 - e^{-z}) \geq \frac{\lambda_n}{n} \frac{3}{8n\lambda_n^2} \sum_{k=1}^{n} (\lambda_k - \lambda_A)^2 \geq \frac{3}{8n^2\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_A)^2.$$

Now, given that $(t+s)^2 \leq 2t^2 + 2s^2$, for $t, s \in \mathbb{R}$, and by noticing that $2n \leq n^2$ for $n \geq 2$, we get

$$\begin{aligned} \sum_{k=1}^{n} (\lambda_k - \lambda_G)^2 &\leq 2 \sum_{k=1}^{n} (\lambda_k - \lambda_A)^2 + 2n \\ &\leq \frac{16}{3} n^2 \lambda_n (\lambda_A - \lambda_G) + 2n\lambda_n(\lambda_A - \lambda_G) \leq 7n^2\lambda_n(\lambda_A - \lambda_G), \end{aligned}$$

where we also used the fact that $\lambda_A - \lambda_G \leq \lambda_A \leq \lambda_n$. □

We will use the following bounds in the proof of Theorem 5.3.

Corollary 5.5. *Let E be a set of finite perimeter with $|E| = |K|$, and let T be the Brenier map of E into K. If $\delta(E) \leq 1$, then*

$$n\,|K|\,\delta(E) \geq \int_{\mathcal{F}E} (1 - ||tr_E(T)||)\,||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1}, \tag{5.5}$$

$$9n^2\,|K|\,\sqrt{\delta(E)} \geq \int_E |\nabla T(x) - \mathrm{id}|\,\mathrm{d}x + |D_s T|\,(E^{(1)}) = |DS|\,(E^{(1)}), \tag{5.6}$$

where $S(x) = T(x) - x$.

Proof. Inequality (5.5) follows immediately from (4.20),

$$\begin{aligned} n\,|K|\,\delta(E) &= P_K(E) - n\,|K|^{1/n}\,|E|^{1/n'} \\ &\geq \int_{\mathcal{F}E} ||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\mathcal{F}E} ||\mathrm{tr}_E(T)||\,||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

From (4.18), (4.19) and (4.7) we deduce that

$$\begin{aligned} |K|\,\delta(E) = \frac{P_K(E)}{n} - |K| &\geq \frac{1}{n} \int_{\mathcal{F}E} ||\mathrm{tr}_E(T)||\,||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1} - \int_E (\det \nabla T)^{1/n} \\ &\geq \frac{1}{n} \int_{\mathcal{F}E} \mathrm{tr}_E(T) \cdot \nu_E\,\mathrm{d}\mathcal{H}^{n-1} - \int_E (\det \nabla T)^{1/n} \\ &\geq \frac{\mathrm{Div}\,T(E^{(1)})}{n} - \int_E (\det \nabla T)^{1/n} \\ &\geq \frac{1}{n} \int_E \mathrm{div}\,T(x) - (\det \nabla T(x))^{1/n}\,\mathrm{d}x + \frac{|D_s T|\,(E^{(1)})}{n}. \end{aligned} \tag{5.7}$$

Adapting the same notation as in the proof of Theorem 4.2, as $\lambda_G = \det \nabla T(x) = 1$ for a.e. $x \in E$, we have

$$\begin{aligned} \int_E |\nabla T(x) - \mathrm{id}|\,\mathrm{d}x = \int_E \sqrt{\sum_{k=1}^{n} (\lambda_k - 1)^2} &\leq \sqrt{||\lambda_n||_{L^1(E)}} \sqrt{\int_E \sum_{k=1}^{n} \frac{(\lambda_k - \lambda_G)^2}{\lambda_n}} \\ &\leq \sqrt{7n^2\,||\lambda_n||_{L^1(E)}\,|K|\,\delta(E)}, \end{aligned} \tag{5.8}$$

where we have first applied the formula for the trace, then introduced λ_n/λ_n in order to apply the Hölder inequality with both exponents 2, and finally used Lemma 5.4 followed by (5.7).

It remains to find an upper bound on $||\lambda_n||_{L^1(E)}$ in order to complete the proof. Since $\lambda_n \geq \lambda_G = 1$ a.e. on E, we can exploit (5.8) together with $|\lambda_n - 1| \leq |\nabla T - \mathrm{id}|$ and $\delta(E) \leq 1$ to obtain

$$\begin{aligned} ||\lambda_n||_{L^1(E)} - |K| &= \int_E |\lambda_n - 1| \leq \int_E |\nabla T - \mathrm{id}| \\ &\leq \sqrt{7}n\sqrt{||\lambda_n||_{L^1(E)}\,|K|} \leq \sqrt{7}n\left(\frac{\varepsilon}{2}\,||\lambda_n||_{L^1(E)} + \frac{1}{2\varepsilon}\,|K|\right), \end{aligned}$$

where we used the inequality $2\sqrt{xy} \leq x + y$ with $x = \varepsilon\,||\lambda_n||_{L^1(E)}$ and $y = |K|/\varepsilon$, for $\varepsilon > 0$. Now, choosing $\varepsilon = 1/(\sqrt{7}n)$ we easily deduce that $||\lambda_n||_{L^1(E)} \leq 8n^2\,|K|$, and with this we obtain

$$\int_E |\nabla T(x) - \mathrm{id}|\,\mathrm{d}x \leq \sqrt{56}n^2\,|K|\,\sqrt{\delta(E)}, \tag{5.9}$$

thanks to (5.8). The result follows from this, together with (5.7), and since $\sqrt{56}+1 \leq 9$. □

5.2 Trace inequalities

Given a set of finite perimeter E in $\mathbb{R}^n$ with $0 < |E| < \infty$, we consider the constant

$$\tau(E) = \inf\left\{\frac{P_K(F)}{\int_{\mathcal{F}F\cap\mathcal{F}E}||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1}} :\ F \subset E,\ 0 < |F| \leq \frac{|E|}{2}\right\},$$

By (1.19), $\tau(E) \geq 1$. The case $\tau(E) = 1$ takes place when E has "undesirable" geometric properties that prevent it from being optimal. For example, that happens when E is not connected or when it has cusps (see Figure 10). Since we want to find a lower bound for (5.6), we can use this idea to hopefully find a Poincaré-Sobolev type inequality, and in order for it to hold, E will have to be somewhat good. That is, $\tau(E) > 1$.

Lemma 5.6. *For every function $f \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and for every set of finite perimeter with $|E| < \infty$, we have*

$$||-Df||_*(E^{(1)}) \geq \frac{m_K}{M_K}\,(\tau(E) - 1)\inf_{c\in\mathbb{R}}\int_{\mathcal{F}E}\mathrm{tr}_E(|f - c|)||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1}. \tag{5.10}$$

Proof. For $t \in \mathbb{R}$, let $F_t = E \cap \{f > t\}$. Setting $c := \sup\{t \in \mathbb{R} :\ |E \setminus F_t| \leq |E|/2\}$, we get $c \in \mathbb{R}$ since $f \in L^\infty(\mathbb{R}^n)$, and it holds true that

$$|F_t| \leq \frac{|E|}{2},\ \forall t \geq c, \quad |E \setminus F_t| \leq \frac{|E|}{2},\ \forall t < c.$$

For convenience, let us introduce the notation

$$u_1(t) = \max\{t - c, 0\}, \quad u_2(t) = \max\{c - t, 0\}, \quad \forall t \in \mathbb{R}.$$

Step 1. We consider the positive and negative parts of $f-c$ and prove a trace inequality for each one. Then, we combine them to find a trace inequality with $|f-c|$.

We start by considering $g = u_1 \circ f = \max\{f-c, 0\}$ and we set $G_s = E \cap \{g > s\}$. By the Anisotropic Coarea Formula (Theorem 4.5) we have that

$$||-Dg||_*(E^{(1)}) = \int_0^\infty \mathrm{d}s \int_{E^{(1)}\cap\mathcal{F}\{g>s\}} \left|\left|\nu_{\{g>s\}}\right|\right|_* \mathrm{d}\mathcal{H}^{n-1} \tag{5.11}$$

Moreover, by (1.12) and (1.15) in Theorem 1.18 we find that $E^{(1)} \cap \mathcal{F}\{g > s\}$ is $\mathcal{H}^{n-1}$-equivalent to $E^{(1)} \cap \mathcal{F}G_s$, since $\mathcal{F}E \cap E^{(1)} = \emptyset$, and that $\nu_{G_s} = \nu_{\{g>s\}}$ at $\mathcal{H}^{n-1}$-a.e. point in $E^{(1)} \cap \mathcal{F}G_s$. Note additionally that thanks to (1.18) and (1.19) we have

$$\begin{aligned}\mathcal{F}G_s &= [\mathcal{F}G_s \cap \mathcal{F}E] \cup [\mathcal{F}G_s \cap E^{(1)}]\\ \nu_{G_s}(x) &= \nu_E(x) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{F}G_s \cap \mathcal{F}E.\end{aligned}$$

Therefore, using the definition of $\tau(E)$, we find

$$\begin{aligned}\int_{E^{(1)}\cap\mathcal{F}\{g>s\}} \left|\left|\nu_{\{g>s\}}\right|\right|_* \mathrm{d}\mathcal{H}^{n-1} &= \int_{E^{(1)}\cap\mathcal{F}G_s} ||\nu_{G_s}||_* \,\mathrm{d}\mathcal{H}^{n-1}\\ &= \int_{\mathcal{F}G_s} ||\nu_{G_s}||_* \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\mathcal{F}E\cap\mathcal{F}G_s} ||\nu_{G_s}||_* \,\mathrm{d}\mathcal{H}^{n-1}\\ &\geq (\tau(E)-1)\int_{\mathcal{F}E\cap\mathcal{F}G_s} ||\nu_{G_s}||_* \,\mathrm{d}\mathcal{H}^{n-1}\\ &= (\tau(E)-1)\int_{\mathcal{F}E\cap\mathcal{F}G_s} ||\nu_E||_* \,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{5.12}$$

We now remark that, by Fubini's Theorem (or layer-cake representation of the function $\mathrm{tr}_E(g)$),

$$\int_{\mathcal{F}E} \mathrm{tr}_E(g)||\nu_E||_* \,\mathrm{d}\mathcal{H}^{n-1} = \int_0^\infty \mathrm{d}s \int_{\mathcal{F}E\cap\mathcal{F}\{\mathrm{tr}_E(g)>s\}} ||\nu_E||_* \,\mathrm{d}\mathcal{H}^{n-1}.$$

We claim that, up to $\mathcal{H}^{n-1}$-null sets,

$$\mathcal{F}E \cap \{\mathrm{tr}_E(g) > s\} \subset \mathcal{F}E \cap \partial_{1/2}G_s.$$

Indeed, by Theorem 1.16 it suffices to show that $\mathcal{H}^{n-1}\left(\mathcal{F}E \cap \{\mathrm{tr}_E(g) > s\} \cap [G_s^{(1)} \cup G_s^{(0)}]\right) = 0$. As $G_s^{(1)} \cap \partial_{1/2}E = \emptyset$, by Theorem 1.16 we find that $\mathcal{H}^{n-1}\left(\mathcal{F}E \cap G_s^{(1)}\right) = 0$. Moreover, if $x \in G_s^{(0)}$,

$$\lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B_r(x)\cap G_s} g(y)\,\mathrm{d}y \leq ||g||_{L^\infty(\mathbb{R}^n)} \lim_{r\to 0} \frac{|B_r(x)\cap G_s|}{|B_r(x)|} = 0.$$

Therefore there can be no $x \in \mathcal{F}E \cap \{\mathrm{tr}_E(g) > s\} \cap G_s^{(0)}$, as otherwise, by (4.9),

$$\begin{aligned}s < \mathrm{tr}_E(g)(x) &= \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)\cap E} g(y)\,\mathrm{d}y\\ &= \lim_{r\to 0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap E\cap\{g\leq s\}} g(y)\,\mathrm{d}y + \frac{1}{|B(x,r)|}\int_{B(x,r)\cap G_s} g(y)\,\mathrm{d}y\right) \leq s,\end{aligned}$$

i.e. we reach a contradiction. In particular, we find that

$$\int_{\mathcal{F}E} \mathrm{tr}_E(g)||\nu_E||_* \,\mathrm{d}\mathcal{H}^{n-1} \leq \int_0^\infty \mathrm{d}s \int_{\mathcal{F}E\cap\mathcal{F}G_s} ||\nu_E||_* \,\mathrm{d}\mathcal{H}^{n-1}, \tag{5.13}$$

and combining (5.11), (5.12) and (5.13) leads to

$$||-D(u_1\circ f)||_*(E^{(1)}) \geq (\tau(E)-1)\int_{\mathcal{F}E} \mathrm{tr}_E(u_1\circ f)||\nu_E||_* \,\mathrm{d}\mathcal{H}^{n-1}. \tag{5.14}$$

We now remark that the choice of c allows us to repeat the above argument with $\max\{c-f,0\}$ in place of $\max\{f-c,0\}$, and we find

$$||D(u_2\circ f)||_*(E^{(1)}) \geq (\tau(E)-1)\int_{\mathcal{F}E} \mathrm{tr}_E(u_2\circ f)||\nu_E||_* \,\mathrm{d}\mathcal{H}^{n-1}. \tag{5.15}$$

Observing now that

$$||y||_* \leq \frac{M_K}{m_K}||-y||_*, \quad \forall y\in\mathbb{R}^n, \tag{5.16}$$

using (5.14), (5.15) and (5.16), and by taking into account the linearity of the trace operator as well as the fact that $u_1(t)+u_2(t)=|t-c|$ for every $t\in\mathbb{R}$, we reach

$$\sum_{k=1}^2 ||-D(u_k\circ f)||_*(E^{(1)}) \geq (\tau(E)-1)\,\frac{m_K}{M_K}\int_{\mathcal{F}E} \mathrm{tr}_E(|f-c|)||\nu_E||_* \,\mathrm{d}\mathcal{H}^{n-1}.$$

Step 2. In order to conclude the proof, we show that for every open set $\Omega\subset\mathbb{R}^n$,

$$\sum_{k=1}^2 ||-D(u_k\circ f)||_*(\Omega) \leq ||-Df||_*(\Omega). \tag{5.17}$$

Indeed, let Ω be fixed. We can find a sequence $\{f_h\}_{h\in\mathbb{N}}\subset C^\infty(\Omega)$ such that $f_h\to f$ in $L^1(\Omega)$, and $\int_\Omega ||-\nabla f_h(x)||_* \,\mathrm{d}x \to ||-Df||_*(\Omega)$ as $h\to\infty$. As $\nabla f_h = 0$ at a.e. $x\in f_h^{-1}(\{c\})$ and $\sum_{k=1}^2 |u_k'(t)| = 1$ for every $t\neq c$, we have

$$\sum_{k=1}^2 \int_\Omega ||-\nabla(u_k\circ f_h)||_* \,\mathrm{d}x \leq \int_\Omega \sum_{k=1}^2 |u_k'(f_h)|\,||-\nabla f_h||_* \,\mathrm{d}x = \int_\Omega ||-\nabla f_h||_* \,\mathrm{d}x.$$

Letting $h\to\infty$, since $u_k\circ f_h\to u_k\circ f$ in $L^1(\Omega)$, by the lower semicontinuity of the anisotropic total variation on open sets we come to (5.17), and therefore the proof is complete. □

5.3 Maximal critical sets

We now show the existence of a maximal critical set for the trace inequality. If we take $\lambda>1$ and define Γ_λ as in the following lemma, Then for small values of $\lambda-1$, every set $F\in\Gamma_\lambda$ is "bad" for the trace inequality. Our goal is to show that for any such λ, there is a "best" set F that contains all the "bad" ones. The idea will then be to remove F from E and reduce the problem, elminating all the parts of E that are "bad" for the trace inequality.

Lemma 5.7 (Existence of a maximal critical set)**.** *Let E be a set of finite perimeter with $0 < |E| < \infty$, and let $\lambda > 1$. If the family of sets*

$$\Gamma_\lambda = \left\{ F \subset E : \ 0 < |F| < \frac{|E|}{2}, \ P_K(F) \leq \lambda \int_{\mathcal{F}F \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1} \right\}$$

is non-empty, then it admits a maximal element with respect to the order relation defined by set inclusion up to sets of measure zero.

Proof. Notice that if we had a maximal element in Γ_λ, then its volume would be larger than that of any other set in that family. It is no use, however, to simply choose the biggest subset of E with volume $|E|/2$, since we need it to satisfy the condition on the perimeter. The way we aim to prove that there exists such a maximal set is by looking at it as the limit of an increasing sequence of sets. Then we apply lower semicontinuity to make sure it is in Γ_λ, and we finally prove it is maximal.

Define a sequence of sets F_h in Γ_λ inductively by first letting F_1 be any element of Γ_λ (which is non-empty). Suppose F_h has been defined for $h \geq 1$, and consider

$$\Gamma_\lambda(h) = \{ F \in \Gamma_\lambda : \ F_h \subset F \} .$$

We then let F_{h+1} be any element of $\Gamma_\lambda(h)$ such that

$$|F_{h+1}| \geq \frac{|F_h| + s_h}{2}, \quad \text{where } s_h = \sup_{F \in \Gamma_\lambda(h)} |F| .$$

The sets $\{F_h\}_{h \in \mathbb{N}}$ form an increasing sequence, with limit F_∞. We claim that $F_\infty \in \Gamma_\lambda$ and that F_∞ is a maximal element in Γ_λ.

Clearly, $|F_\infty| = \sup_{h \in \mathbb{N}} |F_h| \leq |E|/2$. Moreover, by the lower semicontinuity of the anisotropic perimeter we have

$$P_K(F_\infty) \leq \liminf_{h \to \infty} P_K(F_h) \leq \lambda \liminf_{h \to \infty} \int_{\mathcal{F}F_h \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1}. \tag{5.18}$$

Aiming to integrate over $\mathcal{F}F_h \cap \mathcal{F}E$ we notice that $F_h \subset F_{h+1} \subset F_\infty \subset E$, and we find

$$\left(\partial_{1/2} F_h \cap \partial_{1/2} E \right) \subset \left(\partial_{1/2} F_{h+1} \cap \partial_{1/2} E \right) \subset \left(\partial_{1/2} F_\infty \cap \partial_{1/2} E \right), \tag{5.19}$$

hence by (5.18) and Theorem 1.16 it holds true that $F_\infty \in \Gamma_\lambda$. We are left to show that F_∞ is maximal. Indeed, let H be a subset of E, disjoint from F_∞, such that $F_\infty \cup H \in \Gamma_\lambda$. By construction, $F_\infty \cup H \in \Gamma_\lambda(h)$, and

$$s_h \geq |F_\infty \cup H| \geq |F_{h+1}| + |H| \geq \frac{|F_h| + s_h}{2} + |H| ,$$

that is, $|H| \leq (s_h - |F_h|)/2$. However, $s_h - |F_h| \leq 2 \, |F_{h+1} \setminus F_h|$, and this tends to 0 as $h \to \infty$, meaning that $|H| = 0$. This shows the maximality of F_∞. □

5.4 Critical sets in almost optimal sets

Here we show that if E has small isoperimetric deficit, then the sets $F \subset E$ that make $\tau(E) - 1$ small cannot be large in comparison to E and with respect to $\delta(E)$. In

particular, we consider critical sets for the particular value $\lambda = \tau(n)$, independent of E, and show that the deficit of $E \setminus F$ can still be controlled by $\delta(E)$ given that E is almost optimal.

Recall the function Ψ defined in (3.13),

$$\Psi(s) = s^{1/n'} + (1-s)^{1/n'} - 1, \quad s \in [0,1],$$

and notice that

$$\Psi(s) \geq (2 - 2^{1/n'}) s^{1/n'}, \quad s \in [0, 1/2]. \tag{5.20}$$

Set

$$k(n) = \frac{2 - 2^{1/n'}}{3}, \tag{5.21}$$

so that the inequality $3k(n)s^{1/n'} \leq \Psi(s)$ holds true for all $s \in [0,1/2]$, with equality in the cases $s = 0, 1/2$.

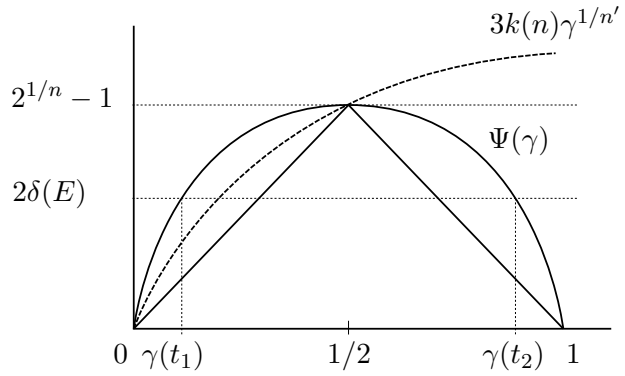


FIGURE 9: The function $\Psi(\gamma) = \gamma^{1/n'} + (1-\gamma)^{1/n'} - 1$.

Lemma 5.8 (Removal of a critical set)**.** *Let E and F be two sets of finite perimeter in $\mathbb{R}^n$ with $F \subset E$ and such that*

$$0 < |F| < \frac{|E|}{2}, \quad P_K(F) \leq \left(1 + \frac{m_K}{M_K} k(n)\right) \int_{\mathcal{F}F \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1}.$$

Then

$$|F| \leq \left(\frac{\delta(E)}{k(n)}\right)^{n'} |E|, \quad P_K(E \setminus F) \leq P_K(E),$$

and in particular, if $\delta(E) \leq k(n)$ then

$$\delta(E \setminus F) \leq \frac{3}{k(n)} \delta(E).$$

Proof. Let us set $\lambda = 1 + (m_K/M_K)k(n)$ and $G = E \setminus F$. Then thanks to (1.18) and (1.19),

$$P_K(F) = \int_{\mathcal{F}E \cap \mathcal{F}F} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1} + \int_{E^{(1)} \cap \mathcal{F}F} ||\nu_F||_* \, \mathrm{d}\mathcal{H}^{n-1}$$
$$P_K(G) = \int_{\mathcal{F}G \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1} + \int_{E^{(1)} \cap \mathcal{F}G} ||\nu_G||_* \, \mathrm{d}\mathcal{H}^{n-1}.$$

An argument by contradiction using the definition of $\partial_{1/2}E$ shows immediately that $\partial_{1/2}F \cap \partial_{1/2}G \cap \partial_{1/2}E = \emptyset$. This implies that

$$P_K(E) = \int_{\mathcal{F}G \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\mathcal{F}F \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1}.$$

Moreover, $\partial_{1/2}F \cap E^{(1)} = \partial_{1/2}G \cap E^{(1)}$, and, by (1.20), $\nu_G = -\nu_F$ at $\mathcal{H}^{n-1}$-a.e. point of $\partial_{1/2}F \cap E^{(1)}$. Gathering these remarks, and taking into account (5.16), we find that

$$P_K(E) \geq P_K(G) + P_K(F) - \left(1 + \frac{M_K}{m_K}\right) \int_{E^{(1)} \cap \mathcal{F}F} ||\nu_F||_* \, \mathrm{d}\mathcal{H}^{n-1}. \tag{5.22}$$

We can visually interpret this as $P_K(G \cup F) = P_K(G) + P_K(F) - 2P_K(E \cap F)$ (see Figure 10). By our assumptions on F together with (1.18),

$$\int_{E^{(1)} \cap \mathcal{F}F} ||\nu_F||_* \, \mathrm{d}\mathcal{H}^{n-1} \leq (\lambda - 1) \int_{\mathcal{F}F \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1} \leq (\lambda - 1) P_K(F).$$

As $(1+(M_K/m_K))(\lambda-1) \leq 2k(n)$, by (5.22) and thanks to the isoperimetric inequality (Theorem 4.2) we derive that

$$\begin{aligned} P_K(E) &\geq P_K(G) + (1 - 2k(n)) \, P_K(F) \\ &\geq n \, |K|^{1/n} \left[|G|^{1/n'} + (1 - 2k(n)) \, |F|^{1/n'} \right]. \end{aligned} \tag{5.23}$$

Let us consider $t = |F| \, / \, |E|$, so that $t \in (0, 1/2]$. By definition of Ψ and of $k(n)$,

$$\delta(E) \geq \Psi(t) - 2k(n) t^{1/n'} \geq k(n) t^{1/n'},$$

and it follows that $|F| \leq (\delta(E)/k(n))^{n'} \, |E|$. Since $k(n) \leq 1/2$ we also deduce from (5.23) that $P_K(G) \leq P_K(E)$. Finally, if $\delta(E) \leq k(n)$, as

$$t \leq \min \left\{ \frac{1}{2}, \frac{\delta(E)}{k(n)} \right\}$$

we find

$$\begin{aligned} \delta(G) &= \frac{P_K(G)}{n \, |K|^{1/n} \, |G|^{1/n'}} - 1 \leq \frac{P_K(E)}{n \, |K|^{1/n} \, |E|^{1/n'} \, (1-t)^{1/n'}} - 1 \\ &\leq \frac{P_K(E)}{n \, |K|^{1/n} \, |E|^{1/n'}} (1 + 2t) - 1 = \delta(E) + 2t \, (\delta(E) + 1) \leq \frac{3}{k(n)} \delta(E). \end{aligned}$$

This completes the proof of the lemma. □

5.5 Reduction to a better set

We now show that an almost optimal set can be replaced by a set that satisfies the trace inequality with a constant bounded from below in terms of n and m_K/M_K only.

Theorem 5.9. *Let E be a set of finite perimeter with $0 < |E| < \infty$, and $\delta(E) \leq k(n)^2/8$. Then there exists $G \subset E$ of finite perimeter and such that*

$$|E \setminus G| \leq \frac{\delta(E)}{k(n)} \, |E| \, , \quad \delta(G) \leq \frac{3}{k(n)} \delta(E), \tag{5.24}$$

and

$$\tau(G) \geq 1 + \frac{m_K}{M_K} k(n). \tag{5.25}$$

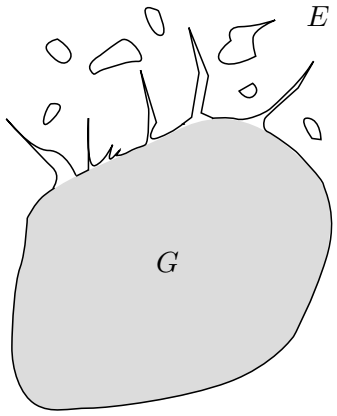


FIGURE 10: The set G is obtained by removing F_∞ from E. If E is almost optimal, then G is as well. We find that $\tau(G) - 1$ is bounded from below in terms on n and K, and the non-trivial trace inequality (5.10) holds on G.

Proof. We consider the family Γ_λ introduced in Lemma 5.7, and take

$$\lambda = 1 + \frac{m_K}{M_K} k(n).$$

Let F_∞ be the maximal set given by Lemma 5.7, and let $G = E \setminus F_\infty$. Since $F_\infty \in \Gamma_\lambda$ and $k(n) < 1$, by Lemma 5.8 we deduce (5.24). Additionally, we have

$$|F_\infty| \leq \frac{\delta(E)}{k(n)} |E| .$$

It remains to show that $\tau(G) \geq \lambda$. Suppose for contradiction that H is a subset of G such that

$$0 < |H| \leq \frac{|G|}{2}, \quad P_K(H) < \lambda \int_{\mathcal{F}H \cap \mathcal{F}G} ||\nu_G||_* \, \mathrm{d}\mathcal{H}^{n-1}. \tag{5.26}$$

We will prove that $F_\infty \cup H \in \Gamma_\lambda$, which contradicts the maximality of F_∞. By Lemma 5.8, and since $G \subset E$, we find

$$|H| \leq \frac{\delta(G)}{k(n)} |G| \leq 3 \frac{\delta(E)}{k(n)^2} |E| ,$$

and likewise, since $\delta(E) \leq k(n)^2/8$ and F_∞ and H are disjoint,

$$|F_\infty \cup H| = |F_\infty| + |H| \leq \frac{\delta(E)}{k(n)} |E| + 3 \frac{\delta(E)}{k(n)^2} |E| \leq 4 \frac{\delta(E)}{k(n)^2} |E| \leq \frac{|E|}{2}.$$

We are therefore left to show that

$$P_K(F_\infty \cup H) \leq \lambda \int_{\mathcal{F}(F_\infty \cup H) \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1},$$

which, given the definition of λ, is equivalent to showing

$$\int_{\mathcal{F}(F_\infty \cup H) \cap E^{(1)}} ||\nu_{F_\infty \cup H}||_* \leq \frac{m_K}{M_K} k(n) \int_{\mathcal{F}(F_\infty \cup H) \cap \mathcal{F}E} ||\nu_E||_* \, \mathrm{d}\mathcal{H}^{n-1}. \tag{5.27}$$

To this end, we remark that by (1.20)

$$\begin{aligned}&\int_{\mathcal{F}(F_\infty\cup H)\cap E^{(1)}} ||\nu_{F_\infty\cup H}||_*\,\mathrm{d}\mathcal{H}^{n-1}\\&\quad=\int_{[\mathcal{F}F_\infty\setminus\mathcal{F}H]\cap E^{(1)}} ||\nu_{F_\infty}||_*\,\mathrm{d}\mathcal{H}^{n-1}+\int_{[\mathcal{F}H\setminus\mathcal{F}F_\infty]\cap E^{(1)}} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}\tag{5.28}$$

Since $\left(\partial_{1/2}H\setminus\partial_{1/2}F_\infty\right)\cap E^{(1)}\subset G^{(1)}$ by Theorem 1.16

$$\begin{aligned}\int_{[\mathcal{F}H\setminus\mathcal{F}F_\infty]\cap E^{(1)}} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1}&\le\int_{\mathcal{F}H\cap G^{(1)}} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1}\\&\le\frac{m_K}{M_K}k(n)\int_{\mathcal{F}H\cap\mathcal{F}G} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}\tag{5.29}$$

where we have also used (5.26). By (1.20), $\nu_{F_\infty}=-\nu_H$ at $\mathcal{H}^{n-1}$-a.e. point of $\mathcal{F}F_\infty\cap\mathcal{F}H$. Therefore, due to (5.16),

$$\int_{\mathcal{F}H\cap\mathcal{F}G\cap E^{(1)}} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1}\le\frac{M_K}{m_K}\int_{\mathcal{F}H\cap\mathcal{F}F_\infty\cap E^{(1)}} ||\nu_{F_\infty}||_*\,\mathrm{d}\mathcal{H}^{n-1}.\tag{5.30}$$

Combining (5.28), (5.29) and (5.30) we find that

$$\begin{aligned}&\int_{\mathcal{F}(F_\infty\cup H)\cap E^{(1)}} ||\nu_{F_\infty\cup H}||_*\,\mathrm{d}\mathcal{H}^{n-1}\\&\quad\le\int_{[\mathcal{F}F_\infty\setminus\mathcal{F}H]\cap E^{(1)}} ||\nu_{F_\infty}||_*\,\mathrm{d}\mathcal{H}^{n-1}\\&\qquad+\frac{m_K}{M_K}k(n)\left(\int_{\mathcal{F}H\cap\mathcal{F}G\cap E^{(1)}} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\mathcal{F}H\cap\mathcal{F}G\cap\mathcal{F}E} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1}\right)\\&\quad\le\int_{\mathcal{F}F_\infty\cap E^{(1)}} ||\nu_{F_\infty}||_*\,\mathrm{d}\mathcal{H}^{n-1}+\frac{m_K}{M_K}k(n)\int_{\mathcal{F}H\cap\mathcal{F}G\cap\mathcal{F}E} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1}\\&\quad\le\frac{m_K}{M_K}k(n)\left(\int_{\mathcal{F}F_\infty\cap\mathcal{F}E} ||\nu_{F_\infty}||_*\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\mathcal{F}H\cap\mathcal{F}G\cap\mathcal{F}E} ||\nu_H||_*\,\mathrm{d}\mathcal{H}^{n-1}\right)\\&\quad\le\frac{m_K}{M_K}k(n)\int_{\mathcal{F}(F_\infty\cup H)\cap\mathcal{F}E} ||\nu_{F_\infty\cup H}||_*\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

where in the last step we have applied (1.20) again. This proves (5.27), thus completing the proof of the theorem. □

5.6 Proof of Theorem 5.3

Before coming to the proof of the theorem, we are left to show one last estimate, which is explained in Figure 8.

Lemma 5.10. *If E is a set of finite perimeter in $\mathbb{R}^n$ with $|E|<\infty$, then*

$$\int_{\mathcal{F}E}|||x||-1|\,||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1}(x)\ge\frac{m_K}{M_K}\,|E\setminus K|\,.\tag{5.31}$$

Proof. Let us set $K_t=tK$ for every $t>0$ since we will use it repeatedly during the proof, and consider the convex function $u:\mathbb{R}^n\longrightarrow[0,\infty)$ defined by $u(x)=||x||$, so

that $K_t = \{u < t\}$ and $u^{-1}(\{t\}) = \partial K_t$. Since u is sub-additive and homogeneous, it can be seen that (5.2) implies

$$\frac{1}{M_K} \leq |\nabla u(x)| \leq \frac{1}{m_K}, \tag{5.32}$$

for a.e. $x \in \mathbb{R}^n$. By an approximation argument on K [Ta, Theorem 3.1], it suffices to prove (5.31) in the case that K is uniformly convex and has smooth boundary. Assuming this, we have that $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and if $x \in \partial K_t$ for some $t > 0$ then

$$\nu_{K_t}(x) = \frac{\nabla u(x)}{|\nabla u(x)|},$$

and due to the convexity of K, the mean curvature H_t of ∂K_t at x satisfies

$$H_t(x) = \operatorname{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right) \geq 0. \tag{5.33}$$

Finally, we notice that $\mathcal{H}^{n-1}(\mathcal{F}E \cap \partial K_t) = 0$ for a.e. $t > 0$ (see [Ma1, Proposition 2.16]), so we can fix $t > 1$, assume that

$$\mathcal{H}^{n-1}(\mathcal{F}E \cap \partial K_t) = 0, \tag{5.34}$$

prove (5.31) for K_t and finish by letting $t \to 1$. Indeed, if we define $u_t(x) = u(x)/t$, $M_{K_t} = tM_K$ and $m_{K_t} = tm_K$, then these satisfy the same relation as (5.32), and $K_t = \{u_t < 1\}$. By (5.32) we have

$$\begin{aligned} |E \setminus K_t| &\leq M_{K_t} \int_{E\setminus K_t} |\nabla u_t| = M_{K_t} \int_{E\setminus K_t} \nabla(u_t - 1) \cdot \frac{\nabla u_t}{|\nabla u_t|} \\ &= M_{K_t} \int_{E\setminus K_t} \operatorname{div}\left((u_t - 1) \cdot \frac{\nabla u_t}{|\nabla u_t|}\right) - (u_t - 1)\operatorname{div}\left(\frac{\nabla u_t}{|\nabla u_t|}\right). \end{aligned} \tag{5.35}$$

Applying the Coarea Formula (Theorem 4.5, with $\psi = (u_t - 1)H_t/|\nabla u_t|$), using the fact that $\{u_t = s\} = \partial K_{st}$, and by (5.33) we find that

$$\int_{E\setminus K_t} (u_t - 1)\operatorname{div}\left(\frac{\nabla u_t}{|\nabla u_t|}\right) = \int_1^\infty (s-1)\,\mathrm{d}s \int_{E\cap\partial K_{st}} \frac{H_t}{|\nabla u_t|}\,\mathrm{d}\mathcal{H}^{n-1} \geq 0.$$

Hence (5.35) and the Divergence Theorem imply

$$|E \setminus K_t| \leq M_{K_t} \int_{\mathcal{F}(E\setminus K_t)} (u_t - 1)\frac{\nabla u_t}{|\nabla u_t|} \cdot \nu_{E\setminus K_t}\,\mathrm{d}\mathcal{H}^{n-1}. \tag{5.36}$$

By (1.13) and by (5.34) we readily see that

$$\nu_{E\setminus K_t}\mathcal{H}^{n-1}\Big|_{\mathcal{F}(E\setminus K_t)} = \nu_E\mathcal{H}^{n-1}\Big|_{(\mathcal{F}E)\setminus K_t} - \nu_{K_t}\mathcal{H}^{n-1}\Big|_{E^{(1)}\cap\partial K_t}.$$

Since $\partial K_t = \{u_t = 1\}$, we conclude from (5.36) that

$$\begin{aligned} |E \setminus K_t| &\leq M_{K_t} \int_{(\mathcal{F}E)\setminus K_t} (u_t - 1)\frac{\nabla u_t}{|\nabla u_t|} \cdot \nu_E\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq M_{K_t} \int_{(\mathcal{F}E)\setminus K_t} ||(u_t - 1)\nu_{K_t}||\,||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq t\frac{M_K}{m_K} \int_{(\mathcal{F}E)\setminus K_t} |u_t - 1|\,||\nu_E||_*\,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

Where we used (4.2) and (5.2). Finally, we complete the proof by letting $t \to 1$. □

Before giving the proof of Theorem 5.3, we introduce the relative asymmetry of two sets E and F, defined as

$$A(E,F) = \inf\left\{ \frac{|E\Delta(x_0 + rF)|}{|E|} : \ x_0 \in \mathbb{R}^n, \ r^n |F| = |E| \right\}$$

because it will appear at the end of the proof.

Proof of Theorem 5.3. Step 1. We prove that

$$A(E) \leq C_0(n,K)\sqrt{\delta(E)},$$

where

$$C_0(n,K) = \frac{181n^3}{(2-2^{1/n'})^{3/2}} \left(\frac{M_K}{m_K}\right)^4.$$

Without loss of generality, we can assume that E is bounded thanks to Theorem 4.6. Since $A(E) \leq 2$, if $\delta(E) \geq k(n)^2/8$, then

$$A(E) \leq 2 \leq \frac{4\sqrt{2}}{k(n)}\sqrt{\delta(E)} \leq C_0(n,K)\sqrt{\delta(E)}.$$

Therefore we assume that $\delta(E) \leq k(n)^2/8$ and apply Theorem 5.9 to find $G \subset E$ for which (5.24) and (5.25) hold true. Unlike with the reduction inequalities in the Euclidean case, G does not originate from a symmetrization of E and its volume is different, so we dilate E and G by the same factor in order to achieve $|G| = |K|$. This operation still allows for (5.24) and (5.25) to hold given the scale invariance of the isoperimetric deficit. We let T be the Brenier map between G and K, and set $S(x) := T(x) - x$. We denote by $S^{(i)}$ the i-th component of S, for $1 \leq i \leq n$. By Corollary 5.5 and by Lemma 5.6, we have, up to a translation,

$$9n^2 |K| \sqrt{\delta(G)} \geq \frac{1}{M_K} \left|\left| -DS^{(i)} \right|\right|_*(G^{(1)}) \geq \frac{m_K^2}{M_K^3} k(n) \int_{\mathcal{F}G} \left|\mathrm{tr}_G(S^{(i)})\right| ||\nu_G||_* \, \mathrm{d}\mathcal{H}^{n-1}.$$

Adding up over $i = 1, ..., n$, and since $\sum_{i=1}^n |y_i| \geq |y| \geq m_K ||y||$, we find that

$$\frac{9n^3}{k(n)} \left(\frac{M_K}{m_K}\right)^3 |K| \sqrt{\delta(G)} \geq \int_{\mathcal{F}G} ||\mathrm{tr}_G(S)|| \, ||\nu_G||_* \, \mathrm{d}\mathcal{H}^{n-1}. \tag{5.37}$$

Once again by Corollary 5.5 we have that

$$n|K| \, \delta(G) \geq \int_{\mathcal{F}G} (1 - ||\mathrm{tr}_G(T)||) \, ||\nu_G||_* \, \mathrm{d}\mathcal{H}^{n-1},$$

so that (5.37) and Lemma 5.10 give

$$\frac{10n^3}{k(n)} \left(\frac{M_K}{m_K}\right)^3 |K| \sqrt{\delta(G)} \geq \int_{\mathcal{F}G} |1 - ||x||| \, ||\nu_G||_* \, \mathrm{d}\mathcal{H}^{n-1} \geq \frac{m_K}{M_K} |G \setminus K| \, .$$

As $|K| \, A(G) \leq |G\Delta K| = 2|G \setminus K|$, we come to

$$A(G) \leq \frac{20n^3}{k(n)} \left(\frac{M_K}{m_K}\right)^4 \sqrt{\delta(G)}.$$

Let us now set $r_E = (|E|/|K|)^{1/n}$, which we recall is different from 1 given that we dilated E, and let $x_G \in \mathbb{R}^n$ be the center of the optimal ball for $A(G)$. Since $|K\Delta(r_E K)| = |E| - |K| = |E \setminus G| = |E\Delta G|$, we obtain

$$\begin{aligned} |E|\, A(E) &\leq |E\Delta(x_G + r_E K)| \leq |E\Delta G| + |G\Delta(x_G + r_E K)| \\ &= |E \setminus G| + |G|\, A(G). \end{aligned}$$

We divide by $|E|$ and take into account both inequalities in (5.24), the definition of $k(n)$, i.e. (5.21), and the fact that $|G| \leq |E|$ to find that

$$\begin{aligned} A(E) &\leq \frac{3\delta(E)}{2 - 2^{1/n'}} + \frac{60n^3}{2 - 2^{1/n'}} \left(\frac{M_K}{m_K}\right)^4 \sqrt{\frac{9\delta(E)}{2 - 2^{1/n'}}} \\ &\leq \frac{181n^3}{(2 - 2^{1/n'})^{3/2}} \left(\frac{M_K}{m_K}\right)^4 \sqrt{\delta(E)}. \end{aligned}$$

This concludes the proof of the first step.

Step 2. We complete the proof of the theorem by showing that

$$A(E) \leq C_0(n)\sqrt{\delta(E)}. \tag{5.38}$$

Here $C_0(n)$ is the constant

$$C_0(n) = \frac{181n^7}{(2 - 2^{1/n'})^{3/2}},$$

and we can assume once again by Theorem 4.6 that E has smooth boundary. By John's Lemma [Fr, Theorem III] there exists an affine map L_0 on $\mathbb{R}^n$ such that

$$B \subset L_0(K) \subset nB, \quad \det L_0 > 0.$$

Thus, we can find $r > 0$ and an affine map L on $\mathbb{R}^n$ such that

$$rB \subset L(K) \subset rnB, \quad \det L = 1,$$

this is, with that choice of $r > 0$ we force $\det L = 1$. We need this in order for L to preserve volumes. By *Step 1* we have that

$$A\left(L(E), L(K)\right) \leq C_0\left(n, L(K)\right) \sqrt{\frac{P_{L(K)}\left(L(E)\right)}{n\,|L(K)|^{1/n}\,|L(E)|^{1/n'}} - 1},$$

where $A\left(L(E), L(K)\right)$ is the relative asymmetry between $L(E)$ and $L(K)$. By construction, $m_{L(K)} \geq r$ and $M_{L(K)} \leq rn$, so

$$\frac{M_{L(K)}}{m_{L(K)}} \leq n,$$

and moreover, as L preserves volumes, we have $A\left(L(E), L(K)\right) = A(E, K) = A(E)$, and therefore

$$A(E) \leq C_0(n) \sqrt{\frac{P_{L(K)}\left(L(E)\right)}{n\,|K|^{1/n}\,|E|^{1/n'}} - 1}.$$

It remains to show that $P_{L(K)}(L(E)) = P_K(E)$. Since E has smooth boundary, we can apply the following generalization of the Minkowski-Steiner formula (see [FiMP, Section 1.2]),

$$\begin{aligned} P_K(E) &= \lim_{\varepsilon \to 0^+} \frac{|E + \varepsilon K| - |E|}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{|L(E + \varepsilon K)| - |L(E)|}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{|L(E) + \varepsilon L(K)| - |L(E)|}{\varepsilon} = P_{L(K)}(L(E)), \end{aligned}$$

which completes the proof of the theorem. □

Recent results due to Segal and Harutyunyan [Se, Hr] show there is room for improvement in the constant $C_0(n)$. In particular, Segal uses an improved arithmetic mean-geometric mean by Alzer [Al] to improve the bound in Corollary 5.5 and show that, for convex sets, $C_0(n)$ can be taken as Cn^7. Harutyunyan then improved this to Cn^6, and Cn^5 for centrally symmetric convex sets, and conjectured that the best constant must be of the form Cn^2. Both approaches are based on the Brunn-Minkowski inequality in connection with the proof of Theorem 5.3 given by Figalli, Maggi and Pratelli [FiMP].

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