THE FABER-KRAHN INEQUALITY FOR THE SHORT TIME FOURIER TRANSFORM

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ABSTRACT. In [Nicola, Tilli, "The Faber-Krahn inequality for the Short-time Fourier transform"] F. Nicola and P. Tilli proved an important result on the energy concentration of functions in the time-frequency plane. We review the original paper and complement it with additional preliminaries, explanations and results in hopes of presenting the topic in a more accessible way.

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1 Time-frequency analysis

The Fourier transform is a powerful tool in harmonic analysis. Its decay carries information about the regularity of a function, it interchanges convolutions with products and modulations with translations, and switches expanding dilations with shrinking ones. While some encoded information about a function is revealed when we look at it through its Fourier transform, there are limitations.

Knowledge of $f(x)$ for all $x \in \mathbb{R}^d$ completely determines all properties of f and gives an explicit expression for the Fourier transform of f,

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \omega},$$

but it is not generally convenient or easy to deduce properties of $\widehat{f}$ by just looking at f. One such hard question is, for example, that of determining whether or not $\widehat{f}$ lies in $L^p(\mathbb{R}^d)$, $p > 0$.

Time-frequency analysis takes a different approach to that of classical Fourier analysis by using the combined information of a function and its Fourier transform at the same time in order to deduce properties about it. As expertly described by K. Gröchenig in [Gr, Section 2.1], there is a very intuitive motivation for how the main tool in time-frequency analysis, the *short-time Fourier transform*, is constructed. We shall explain this intuition in what follows.

Let us think of f as a piece of music. Then $f(x)$ may describe, at time x, the amplitude of the sound coming out of a speaker membrane. By simply observing f we can probably determine some information like the rhythm, tempo or beat of the piece. However, it would be very difficult to determine the melody or the key. On the other hand, the dominating frequency bands of $\widehat{f}$ may give information on what the key is, but not on the rhythm or beat. Additionally, in order to deduce information about $\widehat{f}$, we need to fully know f, i.e., we have to have listened to the whole piece of music before we can identify its main frequencies, while in reality a properly trained listener can almost immediately recognise the key.

The goal of time-frequency analysis is to replicate these mechanisms so as to obtain joint time-frequency representations of signals. These aim to give information about a certain frequency ω of a function f at a specific moment in time x, in a similar flavor to how a musical score determines which notes are playing at a certain time in the piece. One idea by means of which we could materialize such a representation would be taking an "instant Fourier transform" of f at time x, and looking at its value at ω. The only plausible way of doing this, however, would be choosing a small interval around x and computing the Fourier transform of the (possibly smooth) cutoff of f in such interval. But by doing this, we immediately find a clear obstruction in the form of uncertainty principles. The classical uncertainty principle reads as follows.

Theorem 1.1 (Uncertainty principle)**.** *If $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, then*

$$\|(x-a)f\|_2 \left\|(x-b)\widehat{f}\right\|_2 \geq \frac{1}{4\pi} \|f\|_2^2 .$$

Equality holds if and only if f is a multiple of $e^{2\pi ib(x-a)}e^{-\pi(x-a)^2/c}$, for some $a, b \in \mathbb{R}$ and $c > 0$.

It roughly states that the product of the variances of f and $\widehat{f}$ cannot be too small in terms of the L^2 norm of f. This result already provides some intuition on how it is not possible to localize in time without paying a price in frequency, but to fully explain this phenomenon, we present another instance of an uncertainty principle.

Definition 1.2 (ε-concentration)**.** A function $f \in L^2(\mathbb{R}^d)$ is ε-concentrated on a measurable set $T \subset \mathbb{R}^d$ if

$$\left(\int_{T^c} |f(x)|^2 \, \mathrm{d}x\right)^{1/2} \leq \varepsilon \, \|f\|_2 .$$

If ε is small, then most of the energy of f is concentrated on T, and when $\varepsilon = 0$, T is the support of f. The Donoho-Stark uncertainty principle then reads as follows.

Theorem 1.3 (Donoho-Stark uncertainty principle)**.** *Suppose that $f \in L^2(\mathbb{R}^d)$, $f \neq 0$, is ε_T-concentrated on $T \subset \mathbb{R}^d$ and $\widehat{f}$ is ε_Ω-concentrated on $\Omega \subset \mathbb{R}^d$. Then*

$$|T| \, |\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2.$$

Now let us explore the concept of an instantaneous frequency spectrum. Assume that $\|f\|_2 = 1$ and take $x \in \mathbb{R}$. We want to keep track of f during a short time $\delta > 0$, and produce a function of the form $f_x = f \cdot \mathbb{1}_{[x-\delta, x]}$. For any $\varepsilon \geq 0$, f_x is ε-concentrated on $[x - \delta, x]$, and if $\widehat{f}$ were μ-concentrated on Ω for some $\mu > 0$, then the Donoho-Stark uncertainty principle would assert that

$$|\Omega| \geq \frac{(1-\mu)^2}{\delta}.$$

Now, as we let δ get smaller, this implies that we cannot hope to find high concentration of $\widehat{f_x}$ on small sets. Therefore no single frequency band will dominate the rest, making it impossible to think of an instantaneous frequency spectrum. The rough way to express the meaning of uncertainty principles is by saying that f and $\widehat{f}$ cannot be simultaneously concentrated on small sets. In fact, the following result [Gr, Theorem 2.2.3] gives further insight on this phenomenon.

Theorem 1.4. *Suppose that $f \in L^1(\mathbb{R}^d)$ is supported in $T \subset \mathbb{R}^d$, and its Fourier transform is supported in $\Omega \subset \mathbb{R}^d$. If $|T| \, |\Omega| < \infty$, then $f \equiv 0$.*

The ideal time-frequency analysis then finds an obstruction in the form of uncertainty principles. However, we can still build tools that allow us to work with the ideas we have developed. In fact, one of the most basic tools from time-frequency analysis comes precisely from these ideas: the short-time Fourier transform.

1.1 The short-time Fourier transform

As we have discussed already, we may not be able to localize in time in the most precise way without losing frequency resolution, but we can choose a smooth function (called a *window function*) that allows us to look at the frequency spectrum of f over a small window of time without losing too much to the uncertainty. Then, by sliding this window we can generate a representation of f in the time-frequency space. This will be the short-time Fourier transform.

Definition 1.5 (Short-time Fourier transform)**.** Let $f, g \in L^2(\mathbb{R}^d)$ with $g \neq 0$. Then the *short-time Fourier transform* of f with respect to the window g is defined as

$$V_g f(x,\omega) \coloneqq \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\,\mathrm{d}t, \quad \text{for } x,\omega \in \mathbb{R}^d. \tag{1.1}$$

When g is compactly supported and its support is centered around the origin, the function $V_g f$ is the Fourier transform of a local part of f centered at x. This substitutes the previous concept of an "instantaneous frequency spectrum" at x.

The requirement that f and g are in $L^2(\mathbb{R}^d)$ ensures that $V_g f$ is well defined pointwise: using Hölder's inequality we find that the function $f(\cdot)\overline{g(\cdot - x)}$ is in $L^1(\mathbb{R}^d)$, and $V_g f(x,\omega)$ is its Fourier transform at ω. In fact, this argument allows us to define the short-time Fourier transform for $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ whenever p and q are Hölder conjugate exponents. The domain of definition of V can be further extended to the cases when the integral does not immediately make sense. We will not be interested in these cases for now, but a brief discussion on how to treat them can be found in [Gr, p. 41], and a short comment will be necessary in section 3.5.

We will now spend some time discussing a few basic properties of the short-time Fourier transform. Although we will not include the proofs for these statements, they can all be found in [Gr, Section 3.1].

Lemma 1.6. *If $f, g \in L^2(\mathbb{R}^d)$, then $V_g f$ is uniformly continuous on $\mathbb{R}^{2d}$, and*

$$V_g f(x,\omega) = e^{-2\pi i x\cdot\omega} V_{\widehat{g}}\widehat{f}(\omega,-x). \tag{1.2}$$

Identity (1.2) is know as the *fundamental formula of time-frequency analysis*. Proving it is a matter of correctly handling the notation. For this task it is convenient to introduce the most basic (and important) operations in time-frequency analysis: *translations* and *modulations*.

Definition 1.7. For $x, \omega \in \mathbb{R}^d$, we define the *translation by x*

$$T_x f(t) \coloneqq f(t-x),$$

and the *modulation by ω*

$$M_\omega f(t) \coloneqq e^{2\pi i\omega\cdot t} f(t).$$

Operators of the form $M_\omega T_x$ or $T_x M_\omega$ are called *time-frequency shifts*, and they satisfy the following commutation relation

$$T_x M_\omega = e^{-2\pi i x\cdot\omega} M_\omega T_x. \tag{1.3}$$

Time-frequency shifts commute when $x\cdot\omega$ is an integer. They are isometries in $L^p(\mathbb{R}^d)$ for each $1 \leq p \leq \infty$, and the Fourier transform shifts their order in the following way

$$\widehat{T_x f} = M_{-x}\widehat{f}, \quad \widehat{M_\omega f} = T_\omega \widehat{f}.$$

The combination of these last two expressions gives one of the most important formulas in time-frequency analysis,

$$\widehat{T_x M_\omega f} = e^{-2\pi i x\cdot\omega} T_\omega M_{-x}\widehat{f}. \tag{1.4}$$

Using the commutation relation of time-frequency shifts (1.3) we can deduce the following expression relating them to the short-time Fourier transform.

Lemma 1.8. *Whenever $V_g f$ is defined, it holds that*

$$V_g\left(T_u M_\eta f\right)(x,\omega) = e^{-2\pi i u\cdot\omega} V_g f(x-u,\omega-\eta), \quad \text{for } x,u,\omega,\eta \in \mathbb{R}^d. \tag{1.5}$$

1.2 Inversion formula and Lieb's uncertainty principle

We will now present some of the properties that the short-time Fourier transform and the regular Fourier transform share in common. The following discussion can be found in greater detail and complete with proofs in sections 3.2 and 3.3 of [Gr].

An analogue of Parseval's identity can be found for the short-time Fourier transform in the following orthogonality relations [Gr, Theorem 3.2.1].

Theorem 1.9. *Let $f_1, f_2, f_3, f_4 \in L^2(\mathbb{R}^d)$. Then $V_{g_j} f_j \in L^2(\mathbb{R}^{2d})$ for $f = 1, 2$, and*

$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle = \langle f_1, f_2\rangle\, \overline{\langle g_1, g_2\rangle}.$$

We can readily deduce from this result that if $f, g \in L^2(\mathbb{R}^d)$, then

$$\|V_g f\|_2 = \|f\|_2 \|g\|_2,$$

and if $\|g\|_2 = 1$, then V_g is an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$. The natural question of how f can be recovered from $V_g f$ then arises. The fact is that the short-time Fourier transform does indeed have an inversion formula, as is the case with the regular Fourier transform.

Theorem 1.10 (Inversion formula)**.** *Suppose that $g, \gamma \in L^2(\mathbb{R}^d)$ and $\langle g, \gamma\rangle \neq 0$. Then for all $f \in L^2(\mathbb{R}^d)$,*

$$f = \frac{1}{\langle g,\gamma\rangle} \int_{\mathbb{R}^{2d}} V_g f(x,\omega) M_\omega T_x \gamma \,\mathrm{d}\omega\,\mathrm{d}x. \tag{1.6}$$

The proof of the inversion formula can be carried out by first noting that the right-hand side of (1.6) defines a bounded linear functional on $L^2(\mathbb{R}^d)$. Using the Riesz representation theorem we find that this functional, evaluated at $h \in L^2(\mathbb{R}^d)$, is given by the inner product of $\tilde{f} \in L^2(\mathbb{R}^d)$ and h. Finally, it is easy to check that $\langle f, h\rangle = \left\langle \tilde{f}, h\right\rangle$ using Theorem 1.9.

Uncertainty principles exist for the short-time Fourier transform similarly, and we will focus on them later in the paper. For now, let us present a couple of important results in this direction. In doing so, we will follow [Gr, Section 3.3]. The next result is due to E. Lieb [Li].

Theorem 1.11 (Lieb's uncertainty principle)**.** *If $f, g \in L^2(\mathbb{R}^d)$ and $2 \leq p < \infty$, then*

$$\int_{\mathbb{R}^{2d}} |V_g f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega \leq \left(\frac{2}{p}\right)^d \left(\|f\|_2 \|g\|_2\right)^p.$$

Gröchenig showed later [Gr, Theorem 3.3.3] that Lieb's uncertainty yields a new uncertainty principle limiting how small the size of the sets over which $V_g f$ is concentrated can get.

Theorem 1.12. *Suppose that* $\|f\|_2 = \|g\|_2 = 1$. *If* $U \subset \mathbb{R}^{2d}$ *and* $\varepsilon > 0$ *are such that*

$$\int_U |V_g f|^2 \geq 1 - \varepsilon,$$

then

$$|U| \geq (1-\varepsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2d}{p-2}}, \quad \text{for all } p > 2.$$

In particular,

$$|U| \geq \sup_{p>2} (1-\varepsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2d}{p-2}} \geq (1-\varepsilon)^2 2^d.$$

Proof. The result follows by applying Hölder's inequality with exponents $q = p/2$ and $q' = p/(p-2)$ and Lieb's uncertainty principle in the following fashion

$$\begin{aligned} 1 - \varepsilon \leq \int_U |V_g f|^2 &\leq \left(\int_{\mathbb{R}^{2d}} |V_g f|^{2q}\right)^{1/q} \left(\int_{\mathbb{R}^{2d}} \mathbb{1}_U^{q'}\right)^{1/q'} \\ &\leq \left(\frac{2}{p}\right)^{\frac{2d}{p}} \left(\|f\|_2 \|g\|_2\right)^2 |U|^{\frac{p-2}{p}}. \end{aligned}$$

For $p > 2$ we find

$$|U| \geq (1-\varepsilon)^{\frac{p}{p-2}} \left(\frac{p}{2}\right)^{\frac{2d}{p-2}},$$

and when $p = 4$ this reads

$$|U| \geq (1-\varepsilon)^2 2^d. \qquad \square$$

The intuition behind this result can be expressed in a simple way by saying that (the short-time Fourier transform of) a function cannot be concentrated on small sets on the time-frequency plane.

These results, along with the inversion formula for the short-time Fourier transform, illustrate clearly the uncertainty phenomenon that we were discussing earlier, and show how the properties of the window function (g in V_g and γ in the reconstruction of f) affect greatly the resolution of the analysis. Indeed, in order to minimize the loss due to uncertainty, it is of special importance to choose a window function appropriately.

In our case, this choice comes motivated by the characterization of equality cases in Theorem 1.1. Gaussians of the form $T_a M_b \varphi_c(x) = e^{2\pi i b(x-a)} e^{-\pi(x-a)^2/c}$, where

$$\varphi_c(x) \coloneqq e^{-\pi x^2/c},$$

are the minimizers of uncertainty, and for this reason we will focus in these window functions exclusively.

Having discussed the properties of the short-time Fourier transform, we now explain what the aim of this paper is. In [NT], F. Nicola and P. Tilli proved an important result about energy concentration in the time-frequency plane. It provides a sharp bound on how much L^2-norm (energy) the short-time Fourier transform with Gaussian window of a function can accumulate over a measurable set of prescribed Lebesgue measure. It also provides a refinement of Theorem 1.12 in the case of the Gaussian window and a local version of Lieb's uncertainty principle. With this survey we aim to explain in detail the paper of Nicola and Tilli, providing the necessary preliminaries, explanations

and filling in the details. This will be the content discussed in section 3, and the next section will be devoted to learning about one of the tools that is extensively used to prove the results in [NT], the *Bargmann transform*.

2 The Fock space and the Bargmann transform

In light of the previous discussions, we will consider the short-time Fourier transform with a Gaussian window,

$$\varphi(x) = 2^{d/4} e^{-\pi |x|^2}, \quad x \in \mathbb{R}^d, \tag{2.1}$$

modified appropriately so that $\|\varphi\|_2 = 1$. Then V_φ is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$, and it can be expressed as follows

$$\begin{aligned} V_\varphi f(x,\omega) &= 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi (t-x)^2} e^{-2\pi i \omega \cdot t} \, \mathrm{d}t \\ &= 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi t^2} e^{2\pi x \cdot t} e^{-\pi x^2} e^{-2\pi i \omega \cdot t} \, \mathrm{d}t \\ &= 2^{d/4} e^{-\pi i x \cdot \omega} e^{-\frac{\pi}{2}(x^2+\omega^2)} \int_{\mathbb{R}^d} f(t) e^{-\pi t^2} e^{-2\pi t \cdot (x - i\omega)} e^{-\frac{\pi}{2}(x-i\omega)^2} \, \mathrm{d}t. \end{aligned} \tag{2.2}$$

If we turn $(x, \omega) \in \mathbb{R}^{2d}$ into a complex number $z = x + i\omega \in \mathbb{C}^d$, then we can adapt the previous expression for $V_\varphi f$ into an interesting operation. Before presenting it, we introduce some notation. For $t \in \mathbb{R}^d$, will denote by t^2 the product $t \cdot t = |t|^2$, and similarly for a complex number $z = x + i\omega$, i.e. $z^2 = (x + i\omega) \cdot (x + i\omega)$. Its modulus will be $|z|^2 = z \cdot \overline{z} = x^2 + \omega^2$. Lastly, when talking about complex numbers $z, u \in \mathbb{C}^d$, zu will be the product $z \cdot u$.

Definition 2.1 (Bargmann transform)**.** Let $f \in L^2(\mathbb{R}^d)$. We define its *Bargmann transform* on $\mathbb{C}^d$ by

$$Bf(z) \coloneqq 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{2\pi t \cdot z - \pi t^2 - \frac{\pi}{2} z^2} \, \mathrm{d}t. \tag{2.3}$$

The Bargmann transform of a function in $L^2(\mathbb{R}^d)$ actually lies in a particular type of Hilbert space called a *reproducing kernel Hilbert space*. This will be very relevant in the next section, and will be used extensively further on.

Definition 2.2 (Bargmann-Fock space)**.** We denote by $\mathcal{F}(\mathbb{C}^d)$ the *Bargmann-Fock space* of all entire functions on $\mathbb{C}^d$ for which the quantity

$$\|F\|_{\mathcal{F}}^2 \coloneqq \int_{\mathbb{C}^d} |F(z)|^2 \, e^{-\pi |z|^2} \, \mathrm{d}z$$

is finite.

It is easy to check that $\mathcal{F}(\mathbb{C}^d)$ is a Hilbert space when endowed with the product

$$\langle F, G \rangle_{\mathcal{F}} \coloneqq \int_{\mathbb{C}^d} F(z) \overline{G(z)} e^{-\pi |z|^2} \, \mathrm{d}z,$$

since it is the L^2 space of the measure $e^{-\pi |z|^2/2} \, \mathrm{d}z$ on $\mathbb{C}^d$. It turns out that the Bargmann transform and the Bargmann-Fock space are very intimately related. In particular, B allows us to identify $L^2(\mathbb{R}^d)$ with $\mathcal{F}(\mathbb{C}^d)$. To first motivate this, we present the following result.

Proposition 2.3. *If $f \in L^2(\mathbb{R}^d)$ and we write $z = x + i\omega$, then*

$$V_\varphi f(x, -\omega) = e^{i\pi x\cdot\omega} Bf(z) e^{-\pi|z|^2/2}, \tag{2.4}$$

and $B : L^2(\mathbb{R}^d) \longrightarrow \mathcal{F}(\mathbb{C}^d)$ is an isometry, i.e. $\|f\|_2 = \|Bf\|_{\mathcal{F}}$.

Proof. Since $f \in L^2(\mathbb{R}^d)$, the integral in (2.3) converges uniformly for each z in a compact subset of $\mathbb{C}^d$. One can therefore differentiate with respect to z under the integral sign to find that Bf is entire.

Now (2.4) is simply (2.2) with the change of notation, and using it, together with the fact that the short-time Fourier transform is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$, we come to find that

$$\|Bf\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^d} |Bf(z)|^2 \, e^{-\pi|z|^2} \, \mathrm{d}z = \int_{\mathbb{R}^{2d}} |V_\varphi f(x, -\omega)|^2 \, \mathrm{d}x \, \mathrm{d}\omega = \|V_\varphi f\|_2^2 = \|f\|_2^2 .$$

Since Bf is entire, and $\|Bf\|_{\mathcal{F}} = \|f\|_2 < \infty$, it lands within $\mathcal{F}(\mathbb{C}^d)$, and it is also an isometry. In fact, the first argument can be extended to whenever Bf is defined and has polynomial growth, showing that (2.4) holds in that case, and also that Bf is entire. □

The last requirement we need in order to delve into the problem we will discuss for the rest of the paper is some deeper knowledge on the relation between B and $\mathcal{F}(\mathbb{C}^d)$. We will actually show that B is surjective as a map from $L^2(\mathbb{R}^d)$ into $\mathcal{F}(\mathbb{C}^d)$. However, in order to do this we need to explain in detail the structure of $\mathcal{F}(\mathbb{C}^d)$ as a reproducing kernel Hilbert space.

2.1 Reproducing Kernel Hilbert Spaces

The idea behind reproducing kernel Hilbert spaces, or RKHS, is that in these spaces, evaluation functionals are continuous. In particular, this implies that there exists a special function, called a reproducing kernel, that acts by realizing the evaluation functional through the inner product of the Hilbert space.

Definition 2.4 (Reproducing kernel Hilbert space)**.** A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathcal{H}$ where every evaluation functional δ_x, $x \in X$, is continuous and bounded.

The intuition goes as follows. Riesz's representation theorem gives one of the directions without much effort: if evaluation functionals are continuous, then there must exist a function taking a point $x \in X$ as an additional argument, which represents them through the inner product. Similarly, if such a function exists, then the inner product makes it continuous.

Definition 2.5. Let $\mathcal{H}$ be a Hilbert space of complex valued functions over $X \neq \emptyset$. A function $K : X \times X \longrightarrow \mathbb{C}$ is called a reproducing kernel of $\mathcal{H}$ if it satisfies

(i) $\forall x \in X$, the function $K_x = K(\cdot, x) \in \mathcal{H}$. This is called the *representer* at x.

(ii) $\forall x \in X$, $\forall f \in \mathcal{H}$, $\langle f, K(\cdot, x)\rangle = f(x)$. This is known as the *reproducing property.*

A simple remark that follows immediately from the definition is that if $\mathcal{H}$ has a reproducing kernel K, then if $K_x \in \mathcal{H}$ is a function satisfying $K_x(y) = K(y,x)$, it has

$$K(x,y) = \langle K(\cdot,x), K(\cdot,y)\rangle = \langle K_x, K_y\rangle\,, \tag{2.5}$$

and therefore $K(x,y) = \overline{K(y,x)}$. We are now ready to prove the main theorem in this part.

Theorem 2.6. *Let $\mathcal{H}$ be a Hilbert space of functions $f\colon X \longrightarrow \mathbb{C}$. Then the evaluation operators are continuous functionals if and only if $\mathcal{H}$ has a reproducing kernel.*

Proof. Let us assume first that $\mathcal{H}$ has a reproducing kernel. Then, using (2.5)

$$\begin{aligned}|\delta_x f| &= |\langle f, K(\cdot,x)\rangle| \le \|K(\cdot,x)\|\,\|f\| \\ &= \langle K(\cdot,x), K(\cdot,x)\rangle^{1/2}\,\|f\| = K(x,x)^{1/2}\,\|f\|\,.\end{aligned}$$

Hence δ_x is bounded.

Conversely, assume δ_x is bounded. Then the Riesz representation theorem implies that there exists $f_{\delta_x} \in \mathcal{H}$ such that $\delta_x f = \langle f, f_{\delta_x}\rangle$ for every $f \in \mathcal{H}$. Let us define $K(x,y) \coloneqq f_{\delta_x}(y)$ for each pair $x, y \in X$. then K is a reproducing kernel for $\mathcal{H}$. □

Before we leave the topic of RKHS behind, let us explore some of the properties of these spaces. One important feature is the uniqueness of the reproducing kernel. Suppose $\mathcal{H}$ has two reproducing kernels, K_1 and K_2. Then

$$\langle f, K_1(\cdot,x) - K_2(\cdot,x)\rangle = f(x) - f(x) = 0$$

for all $f \in \mathcal{H}$ and all $x \in X$. Hence $\|K_1(\cdot,x) - K_2(\cdot,x)\| = 0$ and $K_1 = K_2$.

Finally, the continuity of evaluation maps also grants pointwise continuity for converging sequences of functions in $\mathcal{H}$. This is, if $(f_n)_{n\ge1} \subset \mathcal{H}$ converge to f in the norm of $\mathcal{H}$, then we can bound in the following way

$$\begin{aligned}|f_n(x) - f(x)| &= |\delta_x(f_n) - \delta_x(f)| = |\delta_x(f_n - f)| \\ &= |\langle f_n - f, K(\cdot,x)\rangle| \le \|K(\cdot,x)\|\,\|f_n - f\|\,.\end{aligned}$$

Since $K(\cdot,x)$ is continuous, its norm is bounded, and the last term converges to 0. We can collect these observations into the following result on the basic properties of Hilbert spaces with reproducing kernels.

Proposition 2.7. *If $\mathcal{H}$ is a RKHS space of functions $f\colon X \longrightarrow \mathbb{C}$ with reproducing kernel K, then*

(i) *K is the unique reproducing kernel of $\mathcal{H}$.*

(ii) *$K(x,y) = \langle K(\cdot,x), K(\cdot,y)\rangle$ for all $x, y \in X$.*

(iii) *$K(x,y) = \overline{K(y,x)}$ for all $x, y \in X$.*

(iv) *If $(f_n)_{n\ge1} \subset \mathcal{H}$ converge to f in $\mathcal{H}$, i.e. $\|f_n - f\| \to 0$ as $n \to \infty$, then $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in \mathcal{H}$.*

2.2 The Bargmann transform

Our goal now is to show that the Bargmann transform is a unitary operator from $L^2(\mathbb{R}^d)$ into $\mathcal{F}(\mathbb{C}^d)$. For this, we only need to prove that its range is dense in $\mathcal{F}(\mathbb{C}^d)$. Our approach to doing so will follow [Gr, Section 3.4].

Proposition 2.8. *The collection of all monomials of the form*

$$e_\alpha(z) = \left(\frac{\pi^{|\alpha|}}{\alpha!}\right)^{1/2} z^\alpha = \prod_{j=1}^d \left(\frac{\pi^{\alpha_j}}{\alpha_j!}\right)^{1/2} z_j^{\alpha_j}$$

for $\alpha = (\alpha_1, ..., \alpha_d)$ with $\alpha_j \geq 0$ forms an orthonormal basis of $\mathcal{F}(\mathbb{C}^d)$.

Proof. We start by proving that the collection $\{e_\alpha : \alpha \geq 0\}$ is an orthonormal system in $\mathcal{F}(\mathbb{C}^d)$. For this, let us start by computing the product of z^α and z^β. To do so, we express them in polar coordinates $z_j = r_j e^{i\theta_j}$ and restrict them to the disc $P_R = \Big\{z \in \mathbb{C}^d : |z_j| \leq R\Big\}$.

$$\begin{aligned}\int_{P_R} z^\alpha \overline{z^\beta} e^{-\pi|z|^2}\,\mathrm{d}z &= \prod_{j=1}^d \int_{|z_j|\leq R} z_j^{\alpha_j}\overline{z_j}^{\beta_j} e^{-\pi|z_j|^2}\,\mathrm{d}z_j \\ &= \prod_{j=1}^d \int_0^R \int_0^{2\pi} r_j^{\alpha_j+\beta_j+1} e^{i(\alpha_j-\beta_j)\theta_j} e^{-\pi r_j^2}\,\mathrm{d}r_j\,\mathrm{d}\theta_j.\end{aligned}$$

If $\alpha \neq \beta$, then this integral is zero for every $R > 0$, and thus $\Big\langle z^\alpha, z^\beta \Big\rangle = 0$. On the other hand, we set

$$\int_{P_R} |z^\alpha|\, e^{-\pi|z|^2}\,\mathrm{d}z = \prod_{j=1}^d \left(2\pi \int_0^R r_j^{2\alpha_j+1} e^{-\pi r_j^2}\,\mathrm{d}r_j\right) = \mu_{\alpha,R}.$$

When $R = \infty$, we can continue with the help of the change of variables $s = \pi r^2$,

$$\mu_{\alpha,\infty} = \prod_{j=1}^d \left(\int_0^\infty \left(\frac{s_j}{\pi}\right)^{\alpha_j} e^{-s}\,\mathrm{d}s\right) = \prod_{j=1}^d \frac{\alpha_j!}{\pi^{\alpha_j}} = \frac{\alpha!}{\pi^{|\alpha|}}.$$

This proves that the set $\{e_\alpha(z) : \alpha \geq 0\}$ is an orthonormal system. We now need to prove that it is complete. Let us consider the power series expansion of F as an entire function,

$$F(z) = \sum_{\alpha\geq 0} c_\alpha z^\alpha.$$

Suppose that $\Big\langle F, e^\beta \Big\rangle = 0$ for all $\beta \geq 0$. Then

$$\langle F, e_\beta\rangle = \lim_{R\to\infty} \left(\frac{\pi^{|\beta|}}{\beta!}\right)^{1/2} \int_{P_R} \sum_{\alpha\geq 0} c_\alpha z^\alpha \overline{z}^\beta e^{-\pi|z|^2}.$$

Since the power series converges uniformly over compact sets, we can interchange the integral and the series, and we obtain

$$0 = \langle F, e_\beta\rangle = \left(\frac{\pi^{|\beta|}}{\beta!}\right)^{1/2} c_\beta \mu_{\beta,\infty}.$$

This implies that $c_\beta = 0$ for all β, and thus $F \equiv 0$. It follows that $\{e_\alpha\}$ is an orthonormal basis of $\mathcal{F}(\mathbb{C}^d)$. □

Knowing this, we can prove that the Bargmann-Fock space has a reproducing kernel.

Theorem 2.9. *The Bargmann-Fock space is a reproducing kernel Hilbert space satisfying*

$$|F(z)| \leq \|F\|_{\mathcal{F}} e^{\pi|z|^2/2}, \quad \forall z \in \mathbb{C}^d. \tag{2.6}$$

The reproducing kernel is $K_\omega(z) = e^{\pi z \cdot \overline{\omega}}$, and equality holds in (2.6) at some point $z_0 \in \mathbb{C}$ if and only if $F = cF_{z_0}$ for some $c \in \mathbb{C}$ with $|c| = \|F\|_{\mathcal{F}}$, where

$$F_{z_0}(z) \coloneqq e^{-\pi|z_0|^2/2} e^{\pi z \cdot \overline{z_0}}, \quad z \in \mathbb{C}.$$

Proof. Given that we can represent $F(z) = \sum_{\alpha \geq 0} \langle F, e_\alpha \rangle e_\alpha(z)$, the Cauchy-Schwarz inequality gives

$$|F(z)| \leq \|F\|_{\mathcal{F}} \left(\sum_{\alpha \geq 0} \frac{\pi^\alpha}{\alpha!} |z^\alpha|^2 \right)^{1/2} = \|F\|_{\mathcal{F}} e^{-\pi|z|^2/2}.$$

Moreover, equality in the Cauchy-Schwarz inequality holds at some $z_0 \in \mathbb{C}$ in terms of the coefficients of F if and only if

$$c_\alpha = \langle F, e_\alpha \rangle = c e^{-\frac{\pi}{2}|z_0|^2} \overline{e_\alpha(z_0)}$$

for some $c \in \mathbb{C}$ with $|c| = \|F\|_{\mathcal{F}}$. This gives

$$F(z) = c e^{-\frac{\pi}{2}|z_0|^2} \sum_{\alpha \geq 0} \overline{e_\alpha(z_0)} e_\alpha(z) = c F_{z_0}(z).$$

Finally, we can obtain $K_\omega(z)$ explicitly by expanding its series representation,

$$K_\omega(z) = \sum_{\alpha \geq 0} \langle K_\omega, e_\alpha \rangle e_\alpha(z) = \sum_{\alpha \geq 0} \overline{e_\alpha(\omega)} e_\alpha(z) = \sum_{\alpha \geq 0} \frac{\pi^{|\alpha|}}{\alpha!} \overline{\omega^\alpha} z^\alpha = e^{\pi \overline{\omega} \cdot z}. \qquad \square$$

Remark. At this point, we remark the fact that that any function F in the Bargmann-Fock space satisfies

$$\lim_{|z| \to \infty} |F(z)|^2 e^{-\pi|z|^2} = 0. \tag{2.7}$$

This can be checked simply using the density of the functions $F_k(z) = z^k$ for $k \geq 0$.

Now that we know that $\mathcal{F}(\mathbb{C}^d)$ is a RKHS, we can prove that the Bargmann transform is a unitary operator.

Theorem 2.10. *The Bargmann transform is a unitary operator from $L^2(\mathbb{R}^d)$ to $\mathcal{F}(\mathbb{C}^d)$.*

Proof. We already know from Proposition 2.3 that B is an isometry. We only need to show that its range is dense in $\mathcal{F}(\mathbb{C}^d)$.

We start by showing that the reproducing kernel lies in the range of B. Let us rewrite Lemma 1.8 in terms of the Bargmann transform. Using (1.3) we first see that

$$V_\varphi(T_u M_{-\eta} \varphi)(x, -\omega) = e^{\pi i (x-u) \cdot (-\eta - \omega)} e^{2\pi i x \cdot \omega} e^{-\pi[(x-u)^2 + (\eta - \omega)^2]/2}.$$

Writing $z = x + i\omega$ and $v = u + i\eta$, we obtain, after using (2.4), that

$$B(T_u M_{-\eta}\varphi) = e^{\pi i u\cdot\eta} e^{-\pi|v|^2/2} e^{\pi \overline{v}\cdot z} = e^{\pi i u\cdot\eta} e^{-\pi|v|^2/2} K_v(z). \tag{2.8}$$

This shows that the reproducing kernel of $\mathcal{F}(\mathbb{C}^d)$ is in the range of B.

We will conclude the proof by showing that any function $F \in \mathcal{F}(\mathbb{C}^d)$ that is orthogonal to the range of B must be zero. Then, the range of B must contain an orthonormal basis, and thus it is dense. Assume that $F \in \mathcal{F}(\mathbb{C}^d)$ has $\langle F, B(f)\rangle = 0$ for all $f \in L^2(\mathbb{R}^d)$. In particular, (2.8) implies that

$$0 = \langle F, B(T_u M_{-\eta}\varphi)\rangle = e^{\pi i u\cdot\eta} e^{-\pi|v|^2/2} \langle F, K_v(z)\rangle = e^{\pi i u\cdot\eta} e^{-\pi|v|^2/2} F(v)$$

for all $v \in \mathbb{C}^d$. Therefore $F \equiv 0$, and the range of B is dense in $\mathcal{F}(\mathbb{C}^d)$. □

3 Energy concentration

The topic of energy concentration of functions on the time-frequency plane is of great interest in mathematics as well as in the applications of time-frequency analysis. We can formulate the necessary notions to treat this issue in terms of the short-time Fourier transform with Gaussian window (STFT).

Indeed, it is natural to interpret $|V_\varphi f(x,\omega)|^2$ as the time-frequency energy density of f. In signal analysis, this relates directly to a particularly useful tool called the *spectrogram* of a signal. Given a window function $g \in L^2(\mathbb{R}^d)$, the spectrogram of $f \in L^2(\mathbb{R}^d)$ is defined as

$$\mathrm{SPEC}_g f(x,\omega) = |V_g f(x,\omega)|^2 .$$

Intuitively, the spectrogram of a signal corresponds to a mathematical sheet music of the signal, and regions with high energy density correspond to dominating frequency bands at certain times. In practice, this means that these regions are the most important when processing signals.

For example, since a spectrogram can only be processed within a bounded region of the time-frequency plane, it is common to use filters that only select the parts of the spectrogram of a signal with high energy density. Techniques like these allow for noise reduction and cancellation effects, and it is through these means that sound recognition software is achieved, allowing the user to, for example, instantly search for songs by sampling them for a few seconds.

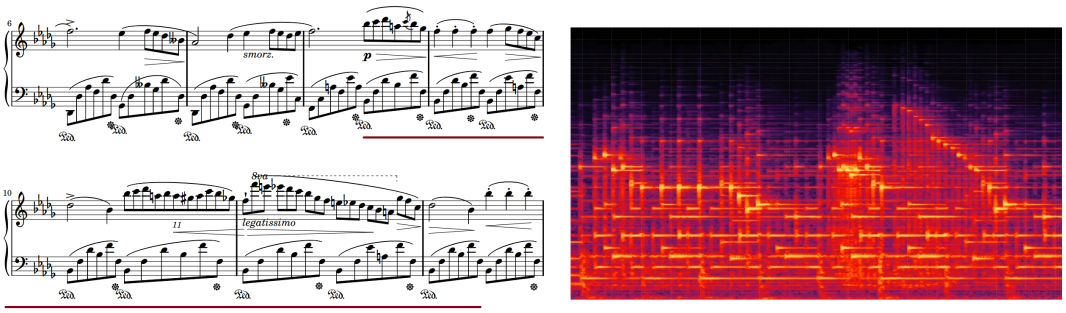


Figure 1: Sheet music and spectrogram of a segment of Chopin's Nocturne op. 9 no. 1 in B flat minor. The underlined part of the sheet corresponds to the image of the spectrogram.

The fraction of energy of $f \in L^2(\mathbb{R}^d)$ captured by a measurable set $\Omega \subset \mathbb{R}^{2d}$ in the time-frequency plane is given by the Rayleigh quotient ([AGR, APR, Da, MR])

$$\Phi_\Omega(f) := \frac{\int_\Omega |V_\varphi f|^2}{\int_{\mathbb{R}^{2d}} |V_\varphi f|^2} = \frac{\left\langle V_\varphi^* \mathbb{1}_\Omega V_\varphi f, f \right\rangle}{\|f\|_2^2}.$$

In particular, when Ω is a set of finite measure, $V_\varphi^* \mathbb{1}_\Omega V_\varphi$ is a compact operator, mapping

bounded subsets of $L^2(\mathbb{R}^d)$ to relatively compact subsets of $L^2(\mathbb{R}^d)$. Its norm is

$$\Phi_\Omega := \max_{f\in L^2(\mathbb{R}^d)\setminus\{0\}} \Phi_\Omega(f) = \max_{f\in L^2(\mathbb{R}^d)\setminus\{0\}} \frac{\left\langle V_\varphi^* \mathbb{1}_\Omega V_\varphi f, f\right\rangle}{\|f\|^2_{L^2(\mathbb{R}^d)}},$$

and it represents the maximum amount of energy that a set Ω can accumulate in the time-frequency plane for any signal $f \in L^2(\mathbb{R}^d)$.

The operator $V_\varphi^* \mathbb{1}_\Omega V_\varphi$ has been thoroughly studied. It appeared first in the works of Berezin [Be] and Daubechies [Da]. In her paper, Daubechies suggested a new geometric approach, treating the time-frequency plane as a single object, rather than two separate spaces. This allowed her to analyze the spectrum of localization operators of radially symmetric sets like spheres. In these cases, the eigenfunctions are Hermite functions, and the eigenvalues can be computed explicitly. The converse was later proved by Abreu and Dörfler [AD], that is, if a Hermite function is the eigenfunction and Ω is simply connected, then it is a ball centered at 0. The asymptotics of the eigenvalues in connection with the measure of Ω have also been studied by many authors [AGR, APR, MR], showing the relations that Ω and the spectrograms of the eigenfunctions share in common.

The subject of energy concentration of functions in the time-frequency plane naturally finds relations with other topics in time-frequency analysis, such as uncertainty principles. Theorem 1.12, which follows from Lieb's uncertainty principle (Theorem 1.11), already shows that domains with high energy concentration cannot be small. A problem that was open until recently, however, was that of finding an upper bound for how much energy a set of prescribed measure can gather in the time-frequency plane. It was the work of Fabio Nicola and Paolo Tilli [NT] in 2021 that provided a solution for said bound, complete with a characterization of the equality cases: balls are the sets that allow for the maximum energy concentration. The aim of this survey is to explain in detail the results in [NT] and the arguments presented in the paper.

3.1 The Faber-Krahn inequality for the STFT

Let us introduce the main results by presenting a brief history of the problem in dimension 1. When Ω is a ball of radius $r > 0$, it follows from Daubechies's work [Da, IVB] that $\Phi_\Omega = 1 - e^{-\pi r^2}$. Later in 2021, Galbis [Ga] proved the inequality $\Phi_\Omega \leq 1 - e^{-|\Omega|}$ for radially symmetric sets $\Omega \subset \mathbb{R}^2$ by using the expression of localization operators in terms of Hermite functions and investigating their eigenvalues.

These two partial results hint at the solution to the problem, which comes in full generality as the main result in the paper of Nicola and Tilli: the Faber-Krahn inequality for the STFT. In dimension 1, it reads as follows [NT, Theorems 1.1, 1.2].

Theorem 3.1 (Faber-Krahn inequality for the STFT)**.** *Given a measurable set $\Omega \subset \mathbb{R}^2$ of positive, finite measure, it holds that*

$$\Phi_\Omega \leq 1 - e^{-|\Omega|}, \tag{3.1}$$

with equality if and only if Ω is equivalent to a ball up to a set of measure zero. Moreover, if Ω is a ball centered at (x_0, ω_0), the only functions f for which equality is achieved are of the form

$$f(x) = cM_{\omega_0}T_{x_0}\varphi(x) = ce^{2\pi i\omega_0 x}\varphi(x - x_0), \quad c \in \mathbb{C}\setminus\{0\}. \tag{3.2}$$

A corollary that is readily drawn from Theorem 3.1 is a sharp uncertainty principle for the STFT, providing a refinement of Theorem 1.12 in the case of the Gaussian window ([NT, Theorem 1.2]). This result solves the long-standing question of finding the optimal bound for the measure of the set in which the STFT of a function is mostly supported. Before this result, the best one in this direction was Theorem 1.12, proved by Gröchenig as a consequence of Lieb's uncertainty principle (Theorem 1.11).

Theorem 3.2 (Sharp uncertainty principle for the STFT)**.** *If $f \in L^2(\mathbb{R}) \setminus \{0\}$ and $\Omega \subset \mathbb{R}^2$ satisfy*

$$\Phi_\Omega(f) \geq 1 - \varepsilon,$$

for some $\varepsilon > 0$, then

$$|\Omega| \geq \log \frac{1}{\varepsilon},$$

with equality if and only if Ω is a ball and f is as in (3.2), with (x_0, ω_0) being the center of Ω.

The reason behind naming Theorem 3.1 the Faber-Krahn inequality for the STFT lies in the similarities it shares with the classic Faber-Krahn inequality, which states that among all open sets Ω of given measure, the ball minimizes the first Dirichlet eigenvalue,

$$\lambda_\Omega \coloneqq \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(z)|^2 \, dz}{\int_\Omega u(z)^2 dz}.$$

If we set $T_\Omega : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ to be the linear operator that to every $u \in H_0^1(\Omega)$ associates its weak solution $T_\Omega u \in H_0^1(\Omega)$ of the problem $-\Delta T_\Omega u = u$ in Ω, then integrating by parts,

$$\int_\Omega u^2 = -\int_\Omega u \Delta T_\Omega u = \int_\Omega \nabla u \cdot \nabla T_\Omega u = \langle T_\Omega u, u \rangle_{H_0^1},$$

and the statement can be stated by claiming that

$$\lambda_\Omega^{-1} \coloneqq \max_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\langle T_\Omega u, u \rangle_{H_0^1}}{\|u\|_{H_0^1}}$$

is maximized, among all open Ω of given measure, by a ball. This establishes a clear similarity between λ_Ω^{-1} and Φ_Ω, the operators T_Ω and $V_\varphi^* \mathbb{1}_\Omega V_\varphi$, and the classic Faber-Krahn inequality and Theorem 3.1.

Although the previous discussion took place in the one dimensional case, we will later see in section 3.3 that these results extend naturally to higher dimensions by just adapting the arguments in dimension one. We will present these in the following section, and later in sections 3.4 and 3.5 we will discuss a few generalizations and applications.

3.2 The result in dimension 1

Recalling Theorem 2.10, where we showed that the Bargmann transform is a unitary operator, we can turn the problem of maximizing energy concentration on the time-frequency plane into a similar problem on the Fock space. Indeed,

$$\Phi_\Omega(f) = \frac{\int_\Omega |V_\varphi f(x,\omega)|^2 \, \mathrm{d}x \, \mathrm{d}\omega}{\|f\|_2^2} = \frac{\int_{\Omega'} |Bf(z)|^2 \, e^{-\pi|z|^2} \, \mathrm{d}z}{\|Bf\|_{\mathcal{F}}^2},$$

where $\Omega' = \{(x, -\omega) : (x, \omega) \in \Omega\}$. Since B is unitary, maximizing $\Phi_\Omega(f)$ over $L^2(\mathbb{R})$ turns out to be the same as solving the problem

$$\Phi_\Omega = \max_{F \in \mathcal{F}(\mathbb{C}) \setminus \{0\}} \frac{\int_\Omega |F(z)|^2 \, e^{-\pi |z|^2} \, \mathrm{d}z}{\|F\|_{\mathcal{F}}^2}. \tag{3.3}$$

The main advantage of this is that the STFT no longer appears in the expression of the problem we aim to solve. Additionally, functions achieving the equality in Theorem 3.1 have a particularly useful correspondence in the Fock space. Indeed, recalling the definition of F_{z_0} in Theorem 2.9, and using (2.8) together with the commutation relation (1.3) yields

$$B(cM_\omega T_x \varphi) = ce^{\pi i x \omega} e^{-\pi |z_0|^2/2} e^{\pi z \overline{z_0}} = c' F_{z_0}, \quad z_0 = x + i\omega, \ c \in \mathbb{C} \setminus \{0\}.$$

Therefore we can state Theorem 3.1 in this new setting as follows ([NT, Theorem 3.1]).

Theorem 3.3. *For every $F \in \mathcal{F}(\mathbb{C}) \setminus \{0\}$ and every set $\Omega \subset \mathbb{R}^2$ of finite measure, we have*

$$\frac{\int_\Omega |F(z)|^2 \, e^{-\pi |z|^2} \, \mathrm{d}z}{\|F\|_{\mathcal{F}}^2} \leq 1 - e^{-|\Omega|}. \tag{3.4}$$

Moreover, equality occurs for some $F \in \mathcal{F}(\mathbb{C})$ and some $\Omega \subset \mathbb{R}^2$ with $0 < |\Omega| < \infty$ if and only if $F = cF_{z_0}$ for some $z_0 \in \mathbb{C}$, and some nonzero $c \in \mathbb{C}$ and Ω is a ball centered at z_0 up to a set of measure zero.

In taking this point of view, given any $F \in \mathcal{F}(\mathbb{C})$, the proof follows an analysis of the integrals of the function

$$u(z) = |F(z)|^2 \, e^{-\pi |z|^2}$$

over its super-level sets $A_t \coloneqq \{u > t\} = \{z \in \mathbb{C} : u(z) > t\}$. The idea is that given a set Ω with positive measure s, if the function u is regular enough we will be able to find a unique level set $A_{t(s)}$ of measure s which accumulates the most mass of u among all sets of equal measure. If we can do this for any arbitrary measure $s > 0$, then we can study the integrals of u over its super-level sets in terms of s, and the problem is reduced to finding a suitable function that bounds these integrals pointwise at every $s > 0$.

We are therefore presented with two main issues. The first one is finding out about how well the super-level sets of u behave, in terms of whether there is uniqueness or not in the sense that we specified before, possibly the regularity of their border, and what the growth of u can tell us about them. The second issue is related to what information we can gather about the aforementioned integrals of u and how the previous analysis can help us find the appropriate bound.

Let us begin to tackle the proof of Theorem 3.3 by introducing a couple of key concepts. These are the *distribution function* of u,

$$\mu(t) \coloneqq |A_t|, \quad 0 < t \leq \max_{\mathbb{C}} u, \tag{3.5}$$

and its inverse, also known as the *decreasing rearrangement*,

$$u^*(s) \coloneqq \sup \{t \geq 0 : \mu(t) > s\}, \quad s \geq 0. \tag{3.6}$$

First, notice that μ is well defined in (3.5) since u is bounded by Theorem 2.9. Additionally, u is non-constant and since $F \in \mathcal{F}(\mathbb{C})$ is holomorphic, u is also real analytic when seen as a function $u : \mathbb{R}^2 \longrightarrow [0, \infty)$. This key property of u already grants some sort of good behaviour on the sets A_t. Indeed, it grants the continuity of μ. Let us briefly show why (a more detailed proof of this can be found in [Mi]).

Proposition 3.4. *If $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a non-constant, real analytic function, then its zero set has Lebesgue measure zero,*

$$|\{z \in \mathbb{R}^n : f(z) = 0\}| = 0.$$

Proof. Suppose $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a non-constant, real analytic function and consider $F_0 := f^{-1}(\{0\})$. If $\nabla f(z) \neq 0$ for some $z = (z', z_n) \in F_0$, then by the Implicit Function Theorem, we can describe F_0 close to z as a graph $z_n = g(z')$ (that is, in a neighbourhood of z'). With this observation one can see that the points of F_0 where ∇f is not 0 will be small. To check it, it is enough to cover it by the family of compact sets G_k,

$$G_k = \left\{ z \in \mathbb{R}^n : |z| \leq k, |\nabla f(z)| \geq \frac{1}{k} \right\}, \quad k \in \mathbb{N}.$$

From here we can choose finitely many neighborhoods of points in each G_k, so we come to find a covering of the set $F_0 \cap \{\nabla f \neq 0\}$ by countably many graphs of dimension $n - 1$. Hence the measure of this set is 0.

To finish the proof, we can use a similar argument after differentiating enough times. This is precisely the reason why we need the analyticity of f. If the function were not analytic or constant, the set of points where all its derivatives vanish could be as big as F_0 in terms of its Lebesgue measure. However, since f is non-constant, by a connectedness argument we find that at some point we will have to find a non-zero derivative. That is, for any $z \in F_0$ with $\nabla f(z) = 0$, there exists $s \in \mathbb{N}^n$ with $D^s f(z) \neq 0$, and $|s| = p + 1$, for $p \in \mathbb{N}$. Therefore, we can trace back one index and find $q \in N^n$ with $|q| = p$ such that

$$\nabla D^q f(z) = (D^{p+e_1} f(z), ..., D^s f(z), ..., D^{p+e_n} f(z)) \neq 0.$$

Now, F_0 can be covered by the sets

$$\left\{ z \in F_0 : D^1 f(z), ..., D^j f(z) = 0, \nabla D^j f(z) \neq 0 \right\}, \quad j \in \mathbb{N}^n.$$

These are all of measure zero and form a countable covering of F_0, which concludes the proof that F_0 has measure zero. □

If we return to the setting we were discussing, $u - t$ is real analytic for all $t > 0$, and we can apply this argument to conclude that

$$|\{u = t\}| = 0 \quad \forall t \geq 0, \tag{3.7}$$

from which the continuity of μ at every $t \in (0, \max u]$ follows. Moreover, the same argument shows that

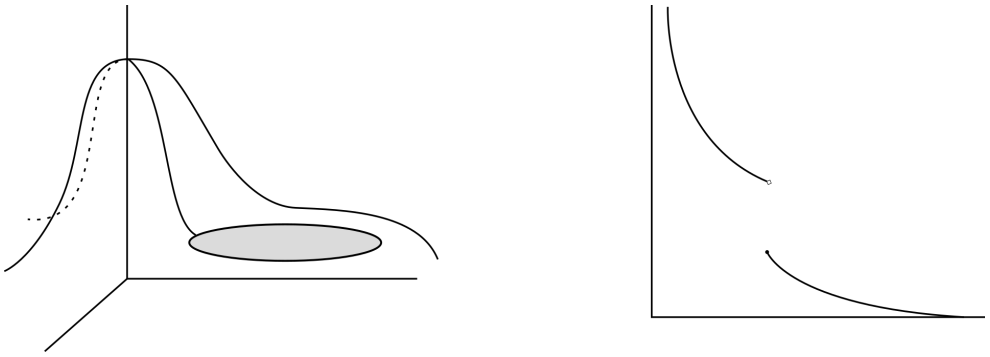
$$\{|\nabla u| = 0\} = 0. \tag{3.8}$$

We can extract some more information about the level sets of u using the Morse-Sard lemma. In fact, this tells us that for a.e. $t \in (0, \max u)$, the super-level set A_t of u is open, bounded and has a smooth boundary,

$$\partial A_t = \{u = t\} \quad \text{for a.e. } t \in (0, \max u). \tag{3.9}$$

Finally, μ is strictly decreasing, which makes u^*, defined in (3.6) as its inverse function, map $[0,\infty)$ into $(0,\max u]$ decreasingly and continuously.

Let us stop here for a moment to motivate the properties of μ and u^* that we can deduce already before presenting them. Condition (3.8) is known as the *coarea regularity* of u in the terminology of [AL, Section 3.6], and it is enough to grant the absolute continuity of μ together with the coarea formula. In fact, seeing $\mu'(t)\,\mathrm{d}t$ as a Radon measure on the compact subintervals of $(0,\max u]$, (3.8) rules out the possibility of it having a singular part. The most intuitive way to see this is to consider a case where u is flat over an open set. This makes μ discontinuous at a point, which gives rise to a singular part of $\mu'(t)\,\mathrm{d}t$ at that point.



Lemma 3.5. *The function μ is absolutely continuous on the compact subintervals of $(0,\max u]$ and*

$$-\mu'(t) = \int_{\{u=t\}} |\nabla u|^{-1}\,\mathrm{d}\mathcal{H}^1, \quad \textit{for a.e. } t \in (0,\max u). \tag{3.10}$$

Similarly, u^* is absolutely continuous on the compact subintervals of $[0,\infty)$ and

$$-(u^*)'(s) = \left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1}\,\mathrm{d}\mathcal{H}^1\right)^{-1}, \quad \textit{for a.e. } s \geq 0. \tag{3.11}$$

Proof. We begin by noticing that u is locally Lipschitz, and therefore we can use the coarea formula at least over open balls centered at the origin. Then, by an approximation argument (using monotone convergence, for instance), we conclude that for any Borel function $h : \mathbb{R}^2 \longrightarrow [0,\infty]$,

$$\int_{\mathbb{R}^2} h(z)\,|\nabla u(z)|\,\mathrm{d}z = \int_0^{\max u} \left(\int_{u=\tau} h\,\mathrm{d}\mathcal{H}^1\right)\mathrm{d}\tau,$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Choosing

$$h(z) = \mathbb{1}_{A_t}(z)\,|\nabla u(z)|^{-1},$$

(with the convention that $0\cdot\infty = 0$ in the integral and $|\nabla u(z)|^{-1}$ is ∞ if z is a critical point of u), we see that $h(z)\,|\nabla u(z)|$ equals $\mathbb{1}_{A_t}(z)$ for a.e. z thanks to (3.8). Then, recalling the definition of μ in (3.5), we find that

$$\mu(t) = \int_t^{\max u} \left(\int_{u=\tau} |\nabla u|^{-1}\,\mathrm{d}\mathcal{H}^1\right)\mathrm{d}\tau, \quad \forall t \in [0,\max u]. \tag{3.12}$$

In particular this expression for μ implies that it is absolutely continuous on the compact subintervals of $(0, \max u]$, and (3.10) follows by straight computation.

Now we address the absolute continuity of u^*. For this, we will check that $(u^*)'$ exists almost everywhere on $(0, \infty)$ and satisfies (3.11), and that u^* maps sets of measure zero into sets of measure zero (known as Luzin's N property). This, together with the fact that it is a function of bounded variation, its continuity and the Banach-Zaretsky Theorem (see [He, Theorem 17]) will prove that it is absolutely continuous.

Let $D \subset (0, \max u]$ be the set where $\mu'(t)$ exists, coincides with (3.10) and is strictly positive, and set $D_0 = (0, \max u] \setminus D$. Notice two things: the integral in (3.10) is strictly positive for all t in $(0, \max u)$ (otherwise the isoperimetric inequality would give $|A_t| = 0$ and $t \geq \max u$), and μ is absolutely continuous, coinciding with (3.10) almost everywhere. This means that $|D_0| = 0$. Letting $\widehat{D} = \mu(D)$ and $\widehat{D_0} = \mu(D_0)$, we find $\left|\widehat{D_0}\right| = 0$ by the absolute continuity of μ, and $\widehat{D} = [0, \infty) \setminus \widehat{D_0}$ since μ is invertible. On the other hand, using (3.10) we come to find that $(u^*)'(s)$ exists for every $s \in \widehat{D}$ and

$$-(u^*)'(s) = -\frac{1}{\mu'\left(\mu^{-1}(s)\right)} = \left(\int_{u=u^*(s)} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^1\right)^{-1}, \quad \forall s \in \widehat{D}.$$

This readily implies (3.11) since $\left|\widehat{D_0}\right| = 0$. Now, since u^* is differentiable everywhere on $\widehat{D}$, it maps sets of measure zero $N \subset \widehat{D}$ into sets of measure zero. Furthermore, since $u^*(\widehat{D_0}) = D_0$ and this set has measure zero, and because $\widehat{D} \cup \widehat{D_0}$ covers the domain of definition $[0, \infty)$ of u^*, it follows that u^* enjoys Luzin's N property. By the aforementioned argument, it is absolutely continuous on every compact interval $[0, a]$. □

Before moving on to studying the integrals of u over its super-level sets, we present a key property that u^* and its derivative satisfy, which will play a pivotal role in the second part of the analysis.

Lemma 3.6. *We have*

$$\left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^1\right)^{-1} \leq u^*(s) \quad \textit{for a.e. } s \geq 0 \tag{3.13}$$

and

$$(u^*)'(s) + u^*(s) \geq 0 \quad \textit{for a.e. } s \geq 0. \tag{3.14}$$

Proof. Let $t = u^*(s)$ and recall that for a.e. $t \in (0, \max u)$, the set A_t has a smooth boundary (3.9). Then we can use Hölder's inequality to obtain

$$\mathcal{H}^1(\{u = t\})^2 = \int_{\{u=t\}} \frac{|\nabla u|^{1/2}}{|\nabla u|^{1/2}} \, \mathrm{d}\mathcal{H}^1 \leq \left(\int_{\{u=t\}} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^1\right) \int_{\{u=t\}} |\nabla u| \, \mathrm{d}\mathcal{H}^1, \tag{3.15}$$

and the isoperimetric inequality reads as

$$4\pi \, |\{u > t\}| \leq \mathcal{H}^1(\{u = t\})^2. \tag{3.16}$$

Combining these two expressions and dividing by t we come to find that

$$\frac{1}{t}\left(\int_{\{u=t\}} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^1\right)^{-1} \leq \frac{\int_{\{u=t\}} \frac{|\nabla u|}{t} \, \mathrm{d}\mathcal{H}^1}{4\pi \, |\{u > t\}|}. \tag{3.17}$$

Notice now that $\{u = t\}$ is the boundary of A_t, and if ν is the outer normal to $\{u = t\}$, then along this set, $|\nabla u| = \nabla u \cdot \nu$. Not only that, since u is equal to t on ∂A_t, the quotient $|\nabla u|/t$ can be interpreted as $-(\nabla u/u)\cdot\nu = -(\nabla \log u)\cdot\nu$, turning the integral on the right-hand side of (3.17) into

$$\int_{\{u=t\}} \frac{|\nabla u|}{t}\,\mathrm{d}\mathcal{H}^1 = -\int_{\partial A_t} (\nabla \log u)\cdot\nu\,\mathrm{d}\mathcal{H}^1 = -\int_{A_t} \Delta \log u(z)\,\mathrm{d}z.$$

Now, since $F(z)$ is a holomorphic function, $\log|F(z)|$ is harmonic (it is the real part of the integral of the holomorphic function $F'(z)/F(z)$) and we obtain

$$\Delta(\log u(z)) = \Delta\left(\log|F(z)|^2 + \log e^{-\pi|z|^2}\right) = \Delta(-\pi\,|z|^2) = -4\pi. \tag{3.18}$$

Using this computation in (3.17) we see that the quotient on the right is 1, and recalling that $t = u^*(s)$, (3.13) follows. Lastly, (3.14) is simply (3.11) put together with (3.13). □

We can now address the second and main part of the analysis. That is, we can study the integrals of u over its super-level sets

$$I(s) = \int_{\{u>u^*(s)\}} u(z)\,\mathrm{d}z, \quad s \in [0,\infty) \tag{3.19}$$

in connection to the function u^*. Intuitively, these integrals gather more energy than integrals over arbitrary sets of the same measure, and given any $\Omega \subset \mathbb{R}^2$ with measure $s > 0$, there is a unique super-level set $A_{u^*(s)}$ of u of measure s. So proving the appropriate bound on $I(s)$ as a function of $s \geq 0$ will allow us to disregard the complexity of Ω and instead use the properties of I to prove Theorem 3.3. Heuristically, these should be that I is slightly more regular than u^* and that its derivatives should satisfy a similar differential equation to (3.14), since by integrating u over $A_{u^*(s)}$ and then differentiating with respect to s one should retrieve the function u^*.

Lemma 3.7. *The function I defined by (3.19) is of class C^1 on $[0,\infty)$, and*

$$I'(s) = u^*(s), \quad s \geq 0. \tag{3.20}$$

Moreover, I' is locally absolutely continuous and

$$I''(s) + I'(s) \geq 0 \quad \textit{for a.e. } s \in [0,\infty). \tag{3.21}$$

Proof. Let $h > 0$ and $s \geq 0$. Then

$$I(s+h) - I(s) = \int_{\{u^*(s+h)<u\leq u^*(s)\}} u(z)\,\mathrm{d}z,$$

and by the definition of u^* in (3.6) and (3.5), $\left|A_{u^*(s)}\right| = s$. This means that the measure of the set over which we are integrating is just

$$|\{u^*(s+h) < u \leq u^*(s)\}| = \left|A_{u^*(s+h)}\right| - \left|A_{u^*(s)}\right| = s + h - s = h.$$

Therefore,

$$u^*(s+h) \leq \frac{I(s+h) - I(s)}{h} \leq u^*(s),$$

and the same inequality holds true for $h < 0$ whenever $s+h > 0$. Since u^* is continuous, it is enough to let $h \to 0$ when $s \geq 0$ and $h \to 0^+$ when $s = 0$ in order to prove (3.20).

The fact that I is C^1 follows from the continuity of u^*, and I' is absolutely continuous on the compact subintervals of $[0, \infty)$ by Lemma 3.5. Finally, $I''(s) = (u^*)'(s)$ and (3.21) follows from (3.14). □

We are now ready to prove Theorem 3.3. As we remarked at the beginning of the section, given the results we have proved so far, the proof reduces to suitably bounding $I(s)$ and characterizing the equality cases.

Proof of Theorem 3.3. We can assume that $\|F\|_{\mathcal{F}} = 1$ by homogeneity, which leaves us with the task of showing that

$$\int_\Omega u(z)\,\mathrm{d}z \leq 1 - e^{-s}$$

for every set $\Omega \subset \mathbb{R}^2$ of measure $s \geq 0$. In the previous discussions we already exposed the beginning of the argument: if $A_{u^*(s)}$ is the super-level set of u that has measure s, then for any Ω of measure s,

$$\int_{A_{u^*(s)}} u(z)\,\mathrm{d}z - \int_\Omega u(z)\,\mathrm{d}z = \int_{A_{u^*(s)}\setminus\Omega} u(z)\,\mathrm{d}z - \int_{\Omega\setminus A_{u^*(s)}} u(z)\,\mathrm{d}z, \tag{3.22}$$

and since $u > u^*(s)$ over the first set, the opposite inequality holds on the second, and they are both of the same measure, it follows that this difference is positive. Therefore, we aim to prove

$$I(s) = \int_{A_{u^*(s)}} u(z)\,\mathrm{d}z \leq 1 - e^{-s}, \quad s \geq 0.$$

By letting $s = -\log\sigma$, this is equivalent to showing

$$G(\sigma) = I(-\log\sigma) \leq 1 - \sigma, \quad \sigma \in (0, 1]. \tag{3.23}$$

Notice that $G(1) = I(0) = 0$, and by monotone convergence,

$$\lim_{\sigma\to 0^+} G(\sigma) = \lim_{s\to\infty} I(s) = \|F\|_{\mathcal{F}} = 1.$$

Then G extends to a continuous function on $[0, 1]$. Moreover, in terms of G, (3.21) means that $G'(\sigma)$ is nondecreasing, which makes G a convex function. Since it agrees with $1 - \sigma$ on $\sigma = 0, 1$, its convexity implies the validity of (3.23).

Let us assume now that equality holds in (3.4) for some $F \in \mathcal{F}(\mathbb{C})$ with $\|F\|_{\mathcal{F}} = 1$ and some $\Omega \subset \mathbb{R}^2$ of measure $s_0 > 0$. Then, in particular $I(s_0) = 1 - e^{-s_0}$, and by setting $\sigma_0 = e^{-s_0}$, we have $G(\sigma_0) = 1 - \sigma_0$. This implies that $G(\sigma)$ coincides with $1 - \sigma$ everywhere by convexity, or equivalently, that $I(s) = 1 - e^{-s}$ for all $s \geq 0$. In particular, $I'(0) = 1$, which by (3.20) means that $\max u = u^*(0) = 1$. Using the characterization of equality cases for (2.6) we deduce that F has to be of the form $F = cF_{z_0}$ for some $z_0 \in \mathbb{C}$ with $|c| = 1$.

We are left to show that Ω is equivalent to a ball centered at z_0. To this end, notice that if equality holds in (3.4), then (3.23) is equal to zero, which means that Ω and $A_{u^*(s_0)}$ coincide up to a set of measure zero. Now, using the expression of F_{z_0} we find that

$$u(z) = |c|^2\,|F_{z_0}|^2\,e^{-1\pi|z|^2} = e^{-\pi|z-z_0|^2}.$$

Therefore u is radially symmetric about z_0 and radially decreasing, so $A_{u^*(s_0)}$ is a ball of measure s_0 centered at z_0.

The last step of the proof is to show that if F is of the form cF_{z_0} and Ω is equivalent to a ball of radius $r > 0$ centered at z_0, then equality in (3.4) holds. This can be done using polar coordinates,

$$\int_\Omega u(z)\,\mathrm{d}z = |c|^2 \int_{\{|z|<r\}} e^{-\pi|z|^2} = 2\pi\,|c|^2 \int_0^r \rho e^{-\pi\rho^2}\,\mathrm{d}\rho = |c|^2\,(1-e^{-\pi r^2}),$$

which completes the proof, since $|c| = \|cF_{z_0}\|_{\mathcal{F}}$. □

3.3 The result in arbitrary dimension

We will devote this section to the generalization of Theorem 3.1 in arbitrary dimension. The approach is exactly the same as the one described in the previous section to prove the result in dimension $d = 1$. Its generalization comes at the cheap cost of taking into account a few constants that depend only on the dimension. To expand on this, let us recall the definitions of a few of the concepts that we will need and give a some details on this new setting.

We will consider now the Gaussian function φ as defined in (2.1), and recall the definition of the STFT for $f \in L^2(\mathbb{R}^d)$ (1.1) with φ for its window function, $V_\varphi f$. We will denote by ω_{2d} the measure of the unit ball in $\mathbb{R}^{2d}$. Additionally, we will need the following (incomplete) version of the gamma function,

$$\gamma(k,s) \coloneqq \int_0^s \tau^{k-1}e^{-\tau}\,\mathrm{d}\tau, \tag{3.24}$$

where $k \geq 1$ is an integer and $s \geq 0$, so

$$\frac{\gamma(k,s)}{(k-1)!} = 1 - e^{-s}\sum_{j=0}^{k-1}\frac{s^j}{j!}. \tag{3.25}$$

Theorem 3.8 (Faber-Krahn inequality for the STFT)**.** *For every measurable set $\Omega \subset \mathbb{R}^{2d}$ of finite measure and for every $f \in L^2(\mathbb{R}^d) \setminus \{0\}$,*

$$\frac{\int_\Omega |V_\varphi f(x,\omega)|^2\,\mathrm{d}x\,\mathrm{d}\omega}{\|f\|_2^2} \leq \frac{\gamma(d, c_\Omega)}{(d-1)!}, \tag{3.26}$$

where $c_\Omega \coloneqq \pi(|\Omega|/\omega_{2d})^{1/d}$. Moreover, equality holds for some f and some Ω with $0 < |\Omega| < \infty$ if and only if Ω is equivalent (up to a set of measure zero) to a ball centered at $(x_0,\omega_0) \in \mathbb{R}^{2d}$, and

$$f(x) = ce^{2\pi i x\cdot\omega_0}\varphi(x-x_0), \quad c \in \mathbb{C}\setminus\{0\}. \tag{3.27}$$

Before moving onto the proof, let us briefly explore the differences we will encounter in trying to follow the approach we used to prove Theorem 3.1. First of all, by virtue of Definitions 2.1 and 2.2 and Theorems 2.9 and 2.10, we can reformulate the problem

in the same way as before using the Fock space $\mathcal{F}(\mathbb{C}^d)$, and knowing it is a RKHS with basis given by the monomials in Proposition 2.8. Then we are left to show

$$\frac{\int_\Omega |F(z)|^2 \, e^{-\pi|z|^2} \, \mathrm{d}z}{\|F\|_{\mathcal{F}}^2} \leq \frac{\gamma(d, c_\Omega)}{(d-1)!} \tag{3.28}$$

when $F \in \mathcal{F}(\mathbb{C}^d)$ and $0 < |\Omega| < \infty$, $\Omega \subset \mathbb{C}^d$, with equality if and only if $F = cF_{z_0}$ and Ω is equivalent to a ball centered at z_0 (up to a set of measure zero).

The definitions of u, μ and u^* are easy to extend to $\mathbb{R}^{2d}$, and the same properties will hold in the case with arbitrary dimension since they all follow from the fact that u is real analytic and Proposition 3.4 is valid in arbitrary dimension. This is, (3.5), (3.6), (3.7), (3.8) and (3.9) will hold in the present case by substituting $\mathbb{R}^2$ by $\mathbb{R}^{2d}$ when necessary, and the discussion between Theorem 3.3 and Lemma 3.5 is valid too in this setting. Furthermore, Lemma 3.5 extends to arbitrary dimension just as easily since the only instance where any specific argument concerning the dimension is used is the coarea formula. In this new version, (3.10) and (3.11) read

$$-\mu'(t) = \int_{\{u=t\}} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^{2d-1}, \quad \text{for a.e. } t \in (0, \max u). \tag{3.29}$$

and

$$-(u^*)'(s) = \left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^{2d-1} \right)^{-1}, \quad \text{for a.e. } s \geq 0. \tag{3.30}$$

The isoperimetric inequality in the proof of Lemma 3.6 gives rise to a few new constants, since this result depends on the dimension. In particular, the proof remains the same in terms of execution, but (3.15) changes in that now we use $\mathcal{H}^{2d-1}$ instead of $\mathcal{H}^1$, (3.16) reads

$$(2d)^2 \omega_{2d}^{1/d} \, |\{u > t\}|^{(2d-1)/d},$$

and instead of $\Delta \log u = -4\pi$, we now have $\Delta \log u = -4\pi d$. These differences combine to replace (3.13) by

$$\left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^{2d-1} \right)^{-1} \leq \pi d^{-1} \omega_{2d}^{-1/d} s^{-1+1/d} u^*(s), \quad a.e. s > 0 \tag{3.31}$$

and (3.14) by

$$(u^*)'(s) + \left(\pi d^{-1} \omega_{2d}^{-1/d} s^{-1+1/d} \right) u^*(s) \geq 0, \quad \text{a.e. } s > 0. \tag{3.32}$$

Using the notation introduced in (3.19), (3.21) in Lemma 3.7 now turns into

$$I''(s) + \left(\pi d^{-1} \omega_{2d}^{-1/d} s^{-1+1/d} \right) I'(s) \geq 0, \quad \text{a.e. } s > 0. \tag{3.33}$$

These are all the changes concerning the results previous to the proof of Theorem 3.3. Now we will give a brief outline of how to keep track of these changes in the argument that proves Theorem 3.8.

Proof of Theorem 3.8. Having rephrased the problem in terms of the Fock space (3.28), we can assume that $\|F\|_{\mathcal{F}} = 1$, and arguing as in the proof of Theorem 3.3 we are led to show

$$I(s) \leq \frac{\gamma(d, \pi(s/\omega_{2d}^{1/d}))}{(d-1)!}, \quad s \geq 0.$$

We can substitute the right-hand side by $1-\sigma$, for $\sigma \in (0,1]$, and setting $G(\sigma) = I(s)$ we are left to show that

$$G(\sigma) \leq 1 - \sigma,$$

as before. The same approach works now, and G extends to a function on $[0,1]$ that coincides with $1-\sigma$ on $\sigma = 0, 1$. The next step is to show that G is convex. To see this, observe that, seeing σ as a function of s,

$$G'(\sigma(s)) = -d!\pi^{-d}\omega_{2d}e^{\pi(s/\omega_{2d})^{1/d}}I'(s).$$

Since the function $e^{\pi(s/\omega_{2d})^{1/d}}I'(s)$ is non decreasing, G' is non decreasing on $(0,1]$, and hence G is convex on $[0,1]$. The proof is then finished with the exact same arguments as that of the one dimensional case. □

A consequence of this result is, as in the one dimensional case, a sharp uncertainty principle for the STFT that generalizes Theorem 3.2. To replace the function $\log(1/\varepsilon)$, which arises as the inverse of e^{-s} in Theorem 3.2, we introduce $\psi_d(\varepsilon)$, $0 < \varepsilon \leq 1$, which is the inverse function of

$$s \mapsto 1 - \frac{\gamma(d,s)}{(d-1)!}, \quad \text{for } s \geq 0.$$

Theorem 3.9 (Sharp uncertainty principle for the STFT)**.** *If $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ and $\Omega \subset \mathbb{R}^{2d}$ satisfy*

$$\int_\Omega |V_\varphi f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega \geq (1-\varepsilon)\,\|f\|_2^2\,,$$

then

$$|\Omega| \geq \omega_{2d}\pi^{-d}\psi_d(\varepsilon)^d,$$

with equality if and only if Ω is a ball and f has the form (3.27)*, where (x_0,ω_0) is the center of the ball.*

3.4 Local Lieb's Uncertainty principle

Two different directions reveal themselves when trying to generalize the techniques developed in the previous sections. The first one concerns a local L^p version of Lieb's uncertainty principle (Theorem 1.11), and we will spend this section describing it. Let us start by considering the problem

$$\sup_{f\in L^2(\mathbb{R})\setminus\{0\}} \frac{\int_\Omega |V_\varphi f(x,\omega)|^p \,\mathrm{d}x\,\mathrm{d}\omega}{\|f\|_2^p}, \tag{3.34}$$

where $\Omega \subset \mathbb{R}^2$ is a measurable set of finite measure and $2 \leq p < \infty$. This problem indeed is a generalization of the previous one, and they coincide when $p = 2$. It turns out that this supremum is in fact attained, and it can be shown by using the direct method of the calculus of variations.

Proposition 3.10. *The supremum in* (3.34) *is attained.*

Proof. In the same spirit as the previous proofs, we can rephrase the problem in the Fock space, obtaining the Rayleigh quotient

$$\frac{\int_\Omega |F(z)|^p \, e^{-p\pi|z|^2/2} \, \mathrm{d}z}{\|F\|_{\mathcal{F}}^p}$$

with $F \in \mathcal{F}(\mathbb{C}) \setminus \{0\}$. Let $(F_n)_{n\geq 1}$ be a maximizing sequence with $\|F_n\|_{\mathcal{F}} = 1$, and set $u_n(z) = |F(z)|^p \, e^{-p\pi|z|^2/2}$. By Theorem 2.9, $u_n(z) \leq \|F_n\|_{\mathcal{F}}^p = 1$, meaning that F_n is bounded uniformly on the compact subsets of $\mathbb{C}$. We can therefore extract a subsequence that converges uniformly on the compact sets of $\mathbb{C}$ to a holomorphic function F, and we will continue naming it F_n. Using Fatou's lemma, we find that $\|F\|_{\mathcal{F}} \leq 1$. Now, given that Ω has finite measure, by the inner regularity of the Lebesgue measure we can find, for any $\varepsilon > 0$, a compact set $K \subset \mathbb{C}$ with $|\Omega \setminus K| < \varepsilon$. This means that

$$\int_{\Omega\setminus K} u_n < \varepsilon, \quad \int_{\Omega\setminus K} |F(z)|^p \, e^{-p|z|^2/2} \, \mathrm{d}z < \varepsilon.$$

Together with the convergence of F_n to F over compact sets, this implies that

$$\lim_{n\to\infty} \int_\Omega u_n(z) \, \mathrm{d}z = \int_\Omega |F(z)^p| \, e^{-p\pi|z|^2/2} \, \mathrm{d}z.$$

Therefore $F \neq 0$ and, since $\|F\|_{\mathcal{F}} \leq 1 = \|F_n\|_{\mathcal{F}}$, we find

$$\lim_{n\to\infty} \frac{\int_\Omega |F_n(z)|^p \, e^{-p\pi|z|^2/2} \, \mathrm{d}z}{\|F_n\|_{\mathcal{F}}^p} \leq \frac{\int_\Omega |F(z)|^p \, e^{-p\pi|z|^2/2} \, \mathrm{d}z}{\|F\|_{\mathcal{F}}^p}.$$

The reverse inequality follows from the fact that F_n is a maximizing sequence. □

Then the local version of Lieb's uncertainty principle corresponds to the problem of maximizing (3.34) over all sets of prescribed measure. The result reads as follows.

Theorem 3.11 (Local Lieb's uncertainty inequality for the STFT)**.** *Let $2 \leq p < \infty$. For every measurable set $\Omega \subset \mathbb{R}^2$ of finite measure and every $f \in L^2(\mathbb{R}) \setminus \{0\}$,*

$$\frac{\int_\Omega |V_\varphi f(x,\omega)|^p \, \mathrm{d}x \, \mathrm{d}\omega}{\|f\|_2^p} \leq \frac{2}{p} \left(1 - e^{-p|\Omega|/2}\right). \tag{3.35}$$

Moreover, equality holds for some function f and some set Ω of finite measure if and only Ω is equivalent to a ball centered at some $(x_0, \omega_0) \in \mathbb{R}^2$ (up to a set of measure zero) and

$$f(x) = ce^{2\pi i x\omega_0} \varphi(x - x_0), \quad c \in \mathbb{C} \setminus \{0\}.$$

The case $p = 2$ reduces the result to Theorem 3.1, and it is possible to recover Theorem 1.11 in dimension $d = 1$ with the Gaussian φ for a window function by using monotone convergence (over increasingly bigger sets converging to $\mathbb{R}^2$) on (3.35).

Proof. The proof follows a similar structure to the proof of Theorem 3.1. Rephrasing the problem in terms of the Fock space, it is enough to show that

$$\frac{\int_\Omega |F(z)|^p \, e^{-p\pi|z|^2/2} \, \mathrm{d}z}{\|F\|_{\mathcal{F}}^p} \leq \frac{2}{p} \left(1 - e^{-p|\Omega|/2}\right)$$

for $F \in \mathcal{F}(\mathbb{C}) \setminus \{0\}$, $0 < |\Omega| < \infty$, and that the equality cases are give by $F = cF_{z_0}$ and the balls Ω of center z_0.

We can assume that $\|F\|_{\mathcal{F}} = 1$ and set $u(z) = |F(z)|^p \, e^{-p\pi|z|^2/2}$. Upon arguing as in Lemmas 3.5 and 3.6, we find that

$$(u^*)'(s) + \frac{p}{2} u^*(s) \geq 0, \quad \text{a.e. } s \geq 0,$$

and with the notation of Lemma 3.7 this turns into

$$I''(s) + \frac{p}{2} I'(s) \geq 0.$$

By arguing as in the proof of Theorem 3.1, the problem is reduced to studying the inequality $I(s) \leq \frac{2}{p}(1 - e^{-ps/2})$, and by changing the variable $s = -\frac{2}{p} \log \sigma$, $\sigma \in (0, 1]$, this turns into

$$G(\sigma) \coloneqq I\left(-\frac{2}{p} \log \sigma\right) \leq \frac{2}{p}(1 - \sigma) \quad \forall \sigma \in (0, 1]. \tag{3.36}$$

The argument is exactly the same as in the proof of Theorem 3.1, with the exception that $G(0) \coloneqq \lim_{\sigma \to 0^+} G(\sigma) = \int_{\mathbb{R}^2} u \leq 2/p$, meaning that at $\sigma = 0$ there can be a strict inequality. When equality happens, we can still use Theorem 2.9 to conclude, with the difference that now in order to deduce the equality $\|F\|_{\mathcal{F}} = \max u$, the computations change slightly,

$$1 = \max u = \max |F(z)|^p \, e^{-p\pi|z|^2/2} = \left(\max |F(z)|^2 \, e^{-\pi|z|^2}\right)^{p/2} \leq \|F\|_{\mathcal{F}}^p = 1.$$

Finally, the converse follows by straight computation, thus proving the result. □

3.5 L^p-concentration estimates

A different direction in which we can extend Theorem 3.1 concerns L^p-concentration estimates. Notice that the problem (3.1) makes sense immediately as the STFT is well defined for L^2 functions. Recalling the discussion following Definition 1.5, it is possible to extend the definition of the STFT to L^p spaces and even spaces where (1.1) does not make sense immediately. In particular, observe that if $f, g \in L^2(\mathbb{R}^d)$, then

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle$$

in the inner product of $L^2(\mathbb{R}^d)$. In the same fashion as how the Fourier transform of a tempered distribution is defined, we can extend the STFT to the class of tempered distributions as long as the window function remains a tempered distribution, i.e., using duality. In particular, this works since translations and modulations of Schwarz functions remain Schwarz functions. With this, if $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, then the STFT of f with respect to g is defined as

$$V_g f(x, \omega) = f(M_\omega T_x g) = \langle f, M_\omega T_x g \rangle \, .$$

Now, when working with tempered distributions, some particularly interesting spaces in time-frequency analysis are *modulation spaces* [Gr, Ch. 11]. They were originally proposed by Hans Georg Feichtinger [Fe], and are now considered the natural spaces

for time-frequency analysis. They roughly consist of those tempered distributions for which the STFT is a function in L^p,

$$M^p(\mathbb{R}) \coloneqq \left\{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{M^p} \coloneqq \|V_\varphi f\|_p < \infty \right\}.$$

Then the STFT is an isometry $V_\varphi : M^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R}^2)$, and the Bargmann transform can be extended $B : M^p(\mathbb{R}) \longrightarrow \mathcal{F}^p(\mathbb{C})$ in the same way (using duality in (2.3)), where $\mathcal{F}^p(\mathbb{C})$ is the space of holomorphic functions $F(z)$ satisfying

$$\|F\|_{\mathcal{F}^p} \coloneqq \left(\int_{\mathbb{C}} |F(z)|^p \, e^{-p\pi|z|^2/2} \, \mathrm{d}z \right)^{1/p} < \infty,$$

while maintaining the validity of (2.4) and so that it is a surjective isometry (see [ST]).

In this context, the methods and arguments described earlier in the proof of Theorem 3.1 work to show the following L^p-concentration estimate.

Theorem 3.12 (L^p-concentration estimates for the STFT)**.** *Let $1 \leq p < \infty$. For every measurable set $\Omega \subset \mathbb{R}^2$ of finite measure and every $f \in M^p(\mathbb{R}) \setminus \{0\}$,*

$$\frac{\int_\Omega |V_\varphi f(x,\omega)|^p \, \mathrm{d}x \, \mathrm{d}\omega}{\int_{\mathbb{R}^2} |V_\varphi f(x,\omega)|^p \, \mathrm{d}x \, \mathrm{d}\omega} \leq 1 - e^{-p|\Omega|/2}. \tag{3.37}$$

Moreover, equality holds for some f and some Ω of finite measure if and only if Ω is equivalent to a ball centered at some $(x_0, \omega_0) \in \mathbb{R}^2$ (up to a set of measure zero), and

$$f(x) = c e^{2\pi i x \omega_0} \varphi(x - x_0). \tag{3.38}$$

The main difference when it comes to the proof of this result with Theorem 3.1 is (2.6). It extends to any $p \in [1, \infty)$ as

$$|F(z)|^p \leq \frac{p}{2} \, \|F\|_{\mathcal{F}^p}^p \, e^{p\pi|z|^2/2},$$

with equality at $z_0 \in \mathbb{C}$ whenever $F = cF_{z_0}$ for some $c \in \mathbb{C} \setminus \{0\}$ [Zh, Theorem 2.7]. Setting $u(z) = |F(z)|^p \, e^{-p\pi|z|^2/2}$, one can follow the previous arguments to reach the desired conclusion. Again, as a consequence we obtain a sharp uncertainty principle for the STFT.

Corollary 3.13. *Let $1 \leq p < \infty$. If for some $\varepsilon > 0$, some measurable set $\Omega \subset \mathbb{R}^2$ of finite measure and some $f \in M^p(\mathbb{R}) \setminus \{0\}$, we have*

$$\int_\Omega |V_\varphi f(x,\omega)|^p \, \mathrm{d}x \, \mathrm{d}\omega \geq \|f\|_{M^p}^p \, (1 - \varepsilon), \tag{3.39}$$

then

$$|\Omega| \geq \frac{2}{p} \log(1/\varepsilon), \tag{3.40}$$

with equality if and only if Ω is equivalent to a ball centered at some $(x_0, \omega_0) \in \mathbb{R}^2$ (up to a set of measure zero), and f is as in (3.38)*.*

Just like in the previous results in this manuscript, Theorems 3.11 and 3.12 can be extended to the multidimensional case in the same way that we described in Section 3.3. As this is a rather mechanical adaptation of the techniques previously demonstrated, we opted to leave it out of this work, since the main ideas have already been displayed in several different contexts.

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