Harmonic Analysis

Lecture notes for a course on harmonic analysis by Prof. Joachim Krieger.

Contents include the L^p theory of Fourier decompositions in $\mathbb{S}^1$ and $\mathbb{R}^n$, the Hilbert transform and Calderón-Zygmund operators.

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Contents

1 Preliminaries in real analysis

We begin by introducing some of the basic concepts and results we will later need throughout the course. This will be the theory of L^p spaces and interpolation of linear operators.

Definition 1.1. Let (X, μ) be a measure space. We denote the space of measurable functions on X by $L^0(X)$. For $1 \leq p \leq \infty$, we define the space of p-integrable functions with respect to μ by

$$L^p(X) := \left\{ f \in L^0(X) \colon \|f\|_{L^p(X,\mu)} < \infty \right\},$$

where

$$\|f\|_{L^p(X,\mu)} := \begin{cases} \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \inf \{ M > 0 \colon |f(x)| \leq M \text{ for } \mu\text{-a.e. } x \} & \text{if } p = \infty. \end{cases}$$

The following lemma is often called the ***Layer-cake formula***. For a function to be in L^p, the measure of its super-level sets must decay to compensate for a $p-1$ power in terms of integrability.

Lemma 1.2 (*Layer-cake formula*). Let $\Phi \colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing function that satisfies $\Phi \in C^1$ and $\Phi(0) = 0$. Then

$$\int_X \Phi(|f|) \, \mathrm{d}\mu = \int_0^\infty \Phi'(\lambda) \mu(\{|f| > \lambda\}) \, \mathrm{d}\lambda.$$

Notice that, in particular, if $\Phi(x) = x^p$,

$$\int_X |f|^p \, \mathrm{d}\mu = p \int_0^\infty \lambda^{p-1} \mu(\{|f| > \lambda\}) \, \mathrm{d}\lambda.$$

Proof of Lemma 1.2. Let us begin by writing

$$\Phi(|f(x)|) = \int_0^{\Phi(|f(x)|)} \mathrm{d}t = \int_0^\infty \mathbb{1}_{\{t < \Phi(|f(x)|)\}} \, \mathrm{d}t.$$

Now, by Fubini's theorem,

$$\int_X \Phi(|f(x)|) \, \mathrm{d}\mu(x) = \int_X \int_0^\infty \mathbb{1}_{\{t < \Phi(|f(x)|)\}} \, \mathrm{d}t \, \mathrm{d}\mu(x) =$$

$$= \int_0^\infty \int_X \mathbb{1}_{\{t<\Phi(|f(x)|)\}} \,\mathrm{d}\mu(x)\,\mathrm{d}t = \int_0^\infty \mu(\{t < \Phi(|f(x)|)\})\,\mathrm{d}t$$

$$= \int_0^\infty \mu(\{\Phi(|f(x)|) > \Phi(\lambda)\})\,\Phi'(\lambda)\,\mathrm{d}\lambda = \int_0^\infty \mu(\{|f(x)| > \lambda\})\,\Phi'(\lambda)\,\mathrm{d}\lambda,$$

where we used the fact that Φ is increasing. □

The following result shows that we can approximate L^p functions by ***continuous, compactly supported functions*** with respect to different topologies in L^p. Also, by looking at such functions as compactly supported, finite Borel measures, they form a dense subset with respect to the weak-* topology.

Proposition 1.3 (*Density of continuous functions*).

(i) For all $p \in [1, \infty)$, $C_c^0(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ is dense with respect to the L^p-norm, that is, if $f \in L^p(\mathbb{R}^n)$, there is a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \|f - \phi_k\|_{L^p(\mathbb{R}^n)} = 0.$$

Moreover, $C^0\left(\mathbb{T}^d\right) \subset L^p\left(\mathbb{T}^d\right)$ is dense with respect to the L^p-norm.

(ii) For $p = \infty$, $C_c^0(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ is dense with respect to the weak-* topology in L^∞, i.e. for all $f \in L^\infty(\mathbb{R}^n)$ there is $\{\phi_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \int \phi_k g = \int fg, \quad \forall g \in L^1(\mathbb{R}^d).$$

Moreover, $C^0\left(\mathbb{T}^d\right) \subset L^p\left(\mathbb{T}^d\right)$ is dense with respect to the weak-* topology in L^∞.

(iii) $C_c^0(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$ is dense with respect to the weak-* topology in $\mathcal{M}(\mathbb{R}^d)$, that is, if $\mu \in \mathcal{M}(\mathbb{R}^d)$, there is $\{\phi_k\}_{k\in\mathbb{N}}$ continuous and compactly supported such that

$$\lim_{k\to\infty} \int \psi\phi_k = \int \psi\,\mathrm{d}\mu \quad \forall \psi \in C_c^0(\mathbb{R}^d).$$

Moreover, $C^0\left(\mathbb{T}^d\right) \subset \mathcal{M}\left(\mathbb{T}^d\right)$ is dense with respect to the weak-* topology in $\mathcal{M}(\mathbb{T}^d)$.

The proof follows by approximating L^p functions and finite Borel measures, respectively, by convolving them with smooth, compactly supported approximate identities ψ_ε. One can also multiply them by some smooth, compactly supported function cutoff of a ball of radius R to obtain the compact support.

Lemma 1.4 (*Translations are continuous in L^p*). Let $p \in [1, \infty)$ and $f \in L^p(X)$, where $X = \mathbb{R}^d$ or $X = \mathbb{T}^d$ with the Lebesgue measure. Then

$$\|f(x-h) - f(x)\|_{L^p(X)} \underset{|h|\to 0}{\longrightarrow} 0.$$

Proof. Fix $\varepsilon > 0$ and take $\phi \in C_c^0(\mathbb{R}^d)$ such that $\|f - \phi\|_{L^p} < \varepsilon$. Using the triangle inequality,

$$\|f(x-h) - f(x)\|_{L^p} \le \|f(x-h) - \phi(x-h)\|_{L^p} + \|\phi(x-h) - \phi(x)\|_{L^p} + \|\phi(x) - f(x)\|_{L^p}$$

$$< 2\varepsilon + \|\phi(x-h) - \phi(x)\|_{L^p}.$$

Since $\phi(x-h)$ converges pointwise to $\phi(x)$ and it is compactly supported, it follows by Lebesgue's dominated convergence that $\|\phi(x-h) - \phi(x)\|_{L^p}$ converges to 0 as $|h| \to 0$. □

Higher power integrability turns into control of lower power norms in the compact setting, which is easily seen to follow from Hölder's inequality. Whenever $1 \leq q < p$,

$$\|f\|_{L^q(\mathbb{S}^1)}^q = \|f^q \mathbb{1}_{\mathbb{S}^1}\|_{L^1(\mathbb{S}^1)} \leq \|f^q\|_{L^{p/q}(\mathbb{S}^1)} \|\mathbb{1}_{\mathbb{S}^1}\|_{L^{(p/q)'}(\mathbb{S}^1)} = \|f\|_{L^p(\mathbb{S}^1)}^q.$$

In the non-compact setting, L^p spaces are related to each other via the following result.

Proposition 1.5. Let $1 \leq p \leq r \leq q \leq \infty$. Then

$$L^p \cap L^q \subset L^r \subset L^p + L^q.$$

More precisely

(i) If $f \in L^p \cap L^q$, then $f \in L^r$ and $\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}$, where

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, \quad \theta \in [0,1].$$

(ii) For all $A > 0$ we can write $f = f_A + f^A$, where

$$f_A = f\mathbb{1}_{\{|f|<A\}}, \quad f^A = f\mathbb{1}_{\{|f|>A\}}$$

and if $f \in L^r$, then

$$f_A \in L^q, \quad \|f_A\|_q^q \leq \frac{q}{r} A^{q-r} \|f\|_r^r,$$
$$f^A \in L^p, \quad \|f^A\|_p^p \leq A^{p-r} \|f\|_r^r.$$

Proof. To show (i), we use Hölder's inequality. Let

$$\theta = \frac{pq - rp}{r(q-p)} \in [0,1]$$

and write $f = f^\theta f^{1-\theta}$. For part (ii), we will use lemma 1.2.

$$\|f_A\|_q^q = \int |f_A|^q \,\mathrm{d}\mu = q\int_0^\infty \lambda^{q-1}\mu(\{|f_A| > \lambda\})\,\mathrm{d}\lambda =$$
$$= q\int_0^A \lambda^{q-1}\mu(\{|f_A| > \lambda\})\,\mathrm{d}\lambda = q\int_0^A \lambda^{q-r}\lambda^{r-1}\mu(\{|f| > \lambda\})\,\mathrm{d}\lambda \leq$$
$$\leq qA^{q-r}\int_0^\infty \lambda^{r-1}\mu(\{|f| > \lambda\})\,\mathrm{d}\lambda = \frac{q}{r}A^{q-r}\|f\|_r^r.$$

For the other inequality,

$$\|f^A\|_p^p = p\int_0^\infty \lambda^{p-1}\mu\big(\{|f^A| > \lambda\}\big)\,\mathrm{d}\lambda$$
$$= \mu(\{|f| > A\})\int_0^A p\lambda^{p-1}\,\mathrm{d}\lambda + p\int_A^\infty \lambda^{p-r}\lambda^{r-1}\mu(\{|f| > \lambda\})\,\mathrm{d}\lambda$$

$$\leq A^p \mu(\{|f| > A\}) + pA^{p-r} \int_A^\infty \lambda^{r-1} \mu(\{|f| > \lambda\}) \, \mathrm{d}\lambda$$
$$= A^p \frac{r}{A^r} \int_0^A \lambda^{r-1} \mu(\{|f| > A\}) + pA^{p-r} \int_A^\infty \lambda^{r-1} \mu(\{|f| > \lambda\}) \, \mathrm{d}\lambda$$
$$\leq rA^{p-r} \int_0^\infty \lambda^{r-1} \mu(\{|f| > \lambda\}) \, \mathrm{d}\lambda = A^{p-r} \|f\|_r^r . \qquad \square$$

1.1 Interpolation theory

The previous result already highlights the fact that whenever a function belongs to two L^p spaces, it belongs to all L^q spaces in between p and q. This showcases the fact that we can relate integrability properties through ***interpolation***. The main idea behind interpolation is being able to find the ***spaces in between*** two given function spaces (like L^p and L^q in this case), so if a function belongs to both endpoints, it will belong to all their intermediate interpolation spaces.

Moreover, we will want to apply this to operators: if T is a bounded operator from A_0 to B_0 and from A_1 to B_1, then we hope that it is also bounded from A_θ to B_θ, for all $\theta \in [0, 1]$ and A_θ, B_θ the spaces in between. We will see two main interpolation results of this type: the ***Marcinkiewicz*** and the ***Riesz-Thorin*** theorems.

Definition 1.6 (*Weak L^p space*). Given $p \in (0, \infty)$ we define $L^{p,\infty}(X)$ as the space of functions $f \in L^0(X)$ such that

$$[f]_{L^{p,\infty}}^p := \sup_{\lambda > 0} \lambda^p \mu(\{|f| > \lambda\}) < \infty.$$

Weak L^p spaces are slightly larger spaces of integrable functions than L^p spaces. During the rest of the chapter, we will denote $\|\cdot\|_{L^p}$ by $\|\cdot\|_p$ and $[\cdot]_{L^{p,\infty}}$ by $[\cdot]_{p,\infty}$.

Lemma 1.7. For all $p \in [1, \infty)$, it holds that $L^p \subset L^{p,\infty}$ and $[f]_{p,\infty} \leq \|f\|_p$.

Proof. The proof is a consequence of the Chebyshev inequality,

$$\mu(\{|f| > \lambda\}) \leq \int_{\{|f| > \lambda\}} (|f| / \lambda)^p \, \mathrm{d}\mu \leq 1/\lambda^p \int_X |f|^p \, \mathrm{d}\mu. \qquad \square$$

<u>*Remark*</u>. These spaces are actually different from L^p, as can be seen when $X = (0, 1)$ and $\mu = \mathcal{L}^1$, since then $f(t) = t^{-1/p} \in L^{p,\infty} \setminus L^p$.

We now have the following result on how weak L^p spaces relate to each other, similar to Proposition 1.5.

Proposition 1.8. Let $1 \leq p < r < q \leq \infty$. Then $L^{p,\infty} \cap L^{q,\infty} \subset L^r \subset L^{p,\infty} + L^{q,\infty}$. More precisely,

(i) If $f \in L^{p,\infty} \cap L^{q,\infty}$, there is $C(p, q, r)$ such that

$$\|f\|_r \leq C[f]_{p,\infty}^\theta [f]_{q,\infty}^{1-\theta}, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

(ii) If we write $f = f_A + f^A$, then

$$[f_A]^q_{q,\infty} \le \|f_A\|^q_q \le \frac{q}{q-r} A^{q-r} [f]^r_{r,\infty},$$
$$[f^A]^p_{p,\infty} \le \|f^A\|^p_p \le \frac{r}{r-p} A^{p-r} [f]^r_{r,\infty}.$$

Proof. For part (i), if $f \in L^{p,\infty} \cap L^{q,\infty}$, then

$$\mu(\{|f| > \lambda\}) \le \min\left\{ \frac{[f]^p_{p,\infty}}{\lambda^p}, \frac{[f]^q_{q,\infty}}{\lambda^q} \right\}.$$

Using this, together with Lemma 1.2, we estimate

$$\begin{aligned}\|f\|^r_r = r\int_0^\infty \lambda^{r-1}\mu(\{|f|>\lambda\})\,\mathrm{d}\lambda &\le r\int_0^M \lambda^{r-p-1}[f]^p_{p,\infty}\,\mathrm{d}\lambda + r\int_M^\infty \lambda^{r-q-1}[f]^q_{q,\infty}\,\mathrm{d}\lambda \\ &= \frac{r}{r-p}[f]^p_{p,\infty}M^{r-p} + \frac{r}{q-r}[f]^q_{q,\infty}M^{q-r}.\end{aligned}$$

Now choose

$$M = \left(\frac{[f]^q_{q,\infty}}{[f]^p_{p,\infty}} \right)^{1/(q-p)}$$

to minimize the expression above. The result follows from this. Part (ii) is left as an exercise. The idea behind this proof is similar to that of the proof of the second part of Proposition 1.5. □

The above shows that L^p is an interpolation space of the weak $L^{q_0,\infty}$ and $L^{q_1,\infty}$ spaces in the interior of the interval $p \in (q_0, q_1)$. Knowing this, we can diret our attention to how it reflects on operators. Moreover, part (ii) already shows how if we obtain a bound for an operator by interpolation, this should become singular at the endpoints p_0, p_1.

Definition 1.9. Let $(X,\mu), (Y,\nu)$ be measure spaces, $D \subset L^0(X)$ a linear subspace and $T : D \longrightarrow L^0(Y,\nu)$.

(i) T is sublinear if $\forall f, g \in D$, $\forall c \in \mathbb{C}$,

$$|T(f+g)| \le |T(f)| + |T(g)|, \quad |T(cf)| = |c|\,|T(f)|.$$

(ii) Let $1 \le p, q \le \infty$. T is Strong(p,q) if $L^p(X) \subset D$ and there exists a constant C such that

$$\|T(f)\|_q \le C\,\|f\|_p \quad \forall f \in L^p(X).$$

(iii) Let $1 \le p, q \le \infty$. T is Weak(p,q) if $L^p(X) \subset D$ and there exists a constant C such that

$$[T(f)]_{q,\infty} \le C\,\|f\|_p \quad \forall f \in L^p(X).$$

Our first interpolation theorem is Marcinkiewicz's interpolation of weak operators. It provides a bound on the norm of sublinear operators of Weak type.

Theorem 1.10 (*Marcinkiewicz*). Let $1 \leq p_0 < p_1 < \infty$ and

$$T : L^{p_0}(X,\mu) + L^{p_1}(X,\mu) \longrightarrow L^0(Y).$$

If T is sublinear, Weak(p_0,p_0) and Weak(p_1,p_1), then T is Strong(p,p) for all $p \in (p_0,p_1)$.

Proof. Let $f \in L^p$, $p \in (p_0,p_1)$ and set $f = f_A + f^A$. Since T is sublinear,

$$|T(f)|(y) = \left|T(f_A + f^A)\right|(y) \leq |T(f_A)|(y) + \left|T(f^A)\right|(y),$$

and given $\lambda > 0$, it is easy to check that

$$\{y\colon\ |T(f)|(y) > 2\lambda\} \subset \{y\colon\ |T(f_A)|(y) > \lambda\} \cup \left\{y\colon\ \left|T(f^A)\right|(y) > \lambda\right\}.$$

With this, we estimate using Lemma 1.2

$$\begin{aligned}\|T(f)\|_p^p &= p\int_0^\infty \tau^{p-1}\nu(\{|T(f)| > \tau\})\,\mathrm{d}\tau = p2^p\int_0^\infty \lambda^{p-1}\nu(\{|T(f)| > 2\lambda\})\,\mathrm{d}\lambda\\ &\leq p2^p\int_0^\infty \lambda^{p-1}\nu(\{|T(f_A)| > \lambda\})\,\mathrm{d}\lambda + p2^p\int_0^\infty \lambda^{p-1}\nu\left(\left\{\left|T(f^A)\right| > \lambda\right\}\right)\mathrm{d}\lambda.\end{aligned}$$

Using the fact that f is Weak(p_0,p_0) on the first term and that it is Weak(p_1,p_1) on the second, we obtain for all A and λ,

$$\nu(\{|T(f_A)| \geq \lambda\}) \leq C_1\frac{\|f_A\|_{p_1}^{p_1}}{\lambda^{p_1}}, \quad \nu\left(\left\{\left|T(f^A)\right| \geq \lambda\right\}\right) \leq C_0\frac{\left\|f^A\right\|_{p_0}^{p_0}}{\lambda^{p_0}}.$$

Choosing $A = \lambda$ and using Fubini's theorem, we get

$$\begin{aligned}\|T(f)\|_p^p &\leq C_1p2^p\int_0^\infty \lambda^{p-p_1-1}\|f_A\|_{p_1}^{p_1}\,\mathrm{d}\lambda + C_0p2^p\int_0^\infty \lambda^{p-p_0-1}\left\|f^A\right\|_{p_0}^{p_0}\mathrm{d}\lambda\\ &= C_1p2^p\int_0^\infty \lambda^{p-p_1-1}\int_{\{|f|\leq\lambda\}}|f|^{p_1}\,\mathrm{d}\mu\,\mathrm{d}\lambda +\\ &\qquad\qquad + C_0p2^p\int_0^\infty \lambda^{p-p_0-1}\int_{\{|f|>\lambda\}}|f|^{p_0}\,\mathrm{d}\mu\,\mathrm{d}\lambda\\ &= C_1p2^p\int_X |f|^{p_1}\int_0^\infty \lambda^{p-p_1-1}\mathbb{1}_{\{|f|\leq\lambda\}}\,\mathrm{d}\lambda\,\mathrm{d}\mu +\\ &\qquad\qquad + C_0p2^p\int_X |f|^{p_0}\int_0^\infty \lambda^{p-p_0-1}\mathbb{1}_{\{|f|>\lambda\}}\,\mathrm{d}\lambda\,\mathrm{d}\mu\\ &= C_1p2^p\int_X |f|^{p_1}\frac{|f|^{p-p_1}}{p_1-p}\,\mathrm{d}\mu + C_0p2^p\int_X |f|^{p_0}\frac{|f|^{p-p_0}}{p-p_0}\,\mathrm{d}\mu = \left[\frac{C_1p2^p}{p_1-p} + \frac{C_0p2^p}{p-p_0}\right]\|f\|_p^p.\end{aligned}$$

□

The intuition for splitting $f \in L^p$ into f^A and f_A lies in the fact that the integrability properties at infinity of f will remain when we place a higher norm (an L^{p_1} norm) on it, while the integrability it enjoys over compact sets (i.e. the badness of its singularities and discontinuities) is not worsened when we place a lower norm (an L^{p_0} norm) on it. In particular, the last expression in the proof

$$\|T(f)\|_p \leq 2p^{1/p}\left[\frac{C_1}{p_1-p} + \frac{C_0}{p-p_0}\right]^{1/p}\|f\|_p\,,$$

shows how the bound on $\|T(f)\|_p$ becomes singular as p approaches the endpoint values p_0 and p_1.

Remark. There exists a more general version of this theorem [HuWe] which states that if f is Weak(p_0, q_0) and Weak(p_1, q_1), then it is Strong(p, q), where

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_0}, \quad 0 < \theta < 1.$$

The next interpolation result is the ***Riesz-Thorin interpolation theorem***. The statement is similar to Theorem 1.10 in that it provides boundedness of an operator in the interpolated spaces, but it now deals with Strong operators, thus obtaining non-singular estimates at the endpoints. Moreover, it generalizes the range of integrability that we assumed in Theorem 1.10.

Theorem 1.11 (*Riesz-Thorin*). Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and (X, μ) a measure space. Assume now that $T : \mathcal{A} \longrightarrow L^0(X)$ is a linear operator from the set of ***simple functions*** to the set of ***Borel*** measurable functions on X, and that it satisfies

$$\|T(f)\|_{q_0} \leq A_0 \|f\|_{p_0}$$
$$\|T(f)\|_{q_1} \leq A_1 \|f\|_{p_1}$$

for all $f \in \mathcal{A}$, i.e. T is Strong(p_0, q_0) and Strong(p_1, q_1). Then for all $\theta \in [0, 1]$,

$$\|Tf\|_q \leq A_0^{\theta} A_1^{1-\theta} \|f\|_p,$$

and T is Strong(p, q), where

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}.$$

In order to prove Theorem 1.11, we will need the following lemma, for which we will not provide a proof.

Lemma 1.12 (*Three lines lemma*). Let f be a holomorphic function on Int S, where $S = \{z \in \mathbb{C} \colon 0 \leq \text{Re}(z) \leq 1\}$, that is bounded on S and extends continuously to $\overline{S}$ with uniform bounds $|f||_{\text{Re}(z)=0} \leq B_0$ and $|f||_{\text{Re}(z)=1} \leq B_1$. Then,

$$|f(z)| \leq B_0^{1-\theta} B_1^{\theta}, \quad \text{for Re}(z) = \theta \in [0, 1].$$

We are now ready to prove the Riesz-Thorin interpolation result.

Proof of Theorem 1.11. The driving idea in the proof is to consider the duality relation

$$\|T(f)\|_q = \sup_{\|g\|_{q'} \leq 1} \left| \int_X T(f) g \, \mathrm{d}\mu \right|, \quad \frac{1}{q} + \frac{1}{q'} = 1$$

in order to apply the lemma to the integral above, after a suitable modification. To this end, by an approximation argument we may consider that both f and g are simple functions

$$f(x) = \sum_{k=1}^{N} a_k e^{i\alpha_k} \mathbb{1}_{A_k}(x), \quad g(x) = \sum_{l=1}^{M} b_l e^{i\beta_l} \mathbb{1}_{B_l}(x),$$

where $A_k, B_l \subset X$ are two collections of measurable, disjoint sets, and $a_k, \alpha_k, b_l, \beta_l \in \mathbb{R}_+$. We will also assume $1 < p, q < \infty$, and leave the rest of the cases as an exercise.

To apply the three lines lemma, we introduce two holomorphic functions

$$f_z(x) = \sum_{k=1}^{M} a_k^{P(z)} e^{i\alpha_k} \mathbb{1}_{A_k}(x), \quad g_z(x) = \sum_{l=1}^{M} b_l^{Q(z)} e^{i\beta_l} \mathbb{1}_{B_l}(x),$$

where

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1}z, \quad Q(z) = \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z.$$

Hence $\int T(f_z)g_z \,\mathrm{d}\mu$ is a holomorphic function which we may expand into finitely many terms where the integral only contains $T(\mathbb{1}_{A_k})\mathbb{1}_{B_k}$.

Assume that $\mathrm{Re}(z) = 0$ and notice that in this case $\left|a_k^{P(z)}\right|^{p_0} = a_k^p$, and therefore

$$\|f_z\|_{p_0}^{p_0} = \int_X \sum_{k=1}^{M} \left|a_k^{P(z)}\right|^{p_0} \mathbb{1}_{A_k}(x) \,\mathrm{d}\mu(x) = \int \sum_{k=1}^{M} a_k^p \mathbb{1}_{A_k}(x) \,\mathrm{d}\mu(x) = \|f\|_p^p.$$

By the same argument, we obtain a similar relation at $\mathrm{Im}(z) = 1$, and for g as well, reaching

$$\begin{aligned} \|f_z\|_{p_0} &= \|f\|_p^{p/p_0}, \quad \|g_z\|_{q'_0} = \|g\|_{q'_0}^{q/q'_0}, \quad \text{at } \mathrm{Re}(z) = 0 \\ \|f_z\|_{p_1} &= \|f\|_p^{p/p_1}, \quad \|g_z\|_{q'} = \|g\|_{q'}^{q/q'_1}, \quad \text{at } \mathrm{Re}(z) = 1. \end{aligned}$$

Using the boundedness of T we also find

$$\begin{aligned} \|T(f_z)\|_{q_0} &\le A_0 \|f_z\|_{p_0} = A_0 \|f\|_{L^p}^{p/p_0}, \quad \text{at } \mathrm{Re}(z) = 0, \\ \|T(f_z)\|_{q_1} &\le A_1 \|f_z\|_{p_1} = A_1 \|f\|_{L^p}^{p/p_1}, \quad \text{at } \mathrm{Re}(z) = 1, \end{aligned}$$

Using Hölder's inequality we find

$$\begin{aligned} \left|\int T(f_z)g_z \,\mathrm{d}x\right| &\le A_0 \|f\|_p^{p/p_0} \|g\|_{q'}^{q'/q'_0}, \quad \text{for } \mathrm{Re}(z) = 0, \\ \left|\int T(f_z)g_z \,\mathrm{d}x\right| &\le A_1 \|f\|_p^{p/p_1} \|g\|_{q'}^{q'/q'_1}, \quad \text{for } \mathrm{Re}(z) = 1. \end{aligned}$$

At this point we use Lemma 1.12 to find that for $\mathrm{Re}(z) = \tilde{\theta} \in [0,1]$, we have

$$\left|\int T(f_z)g_z \,\mathrm{d}x\right| \le A_0^{1-\tilde{\theta}} A_1^{\tilde{\theta}} \|f\|_p^{\frac{p}{p_0}(1-\tilde{\theta}) + \frac{p}{p_1}\tilde{\theta}} \|g\|_{q'}^{\frac{q}{q'_0}(1-\tilde{\theta}) + \frac{q}{q'_1}\tilde{\theta}}. \tag{1.1}$$

Setting $z = \theta = \tilde{\theta}$, where θ is as in the choice of p and q, we find

$$T(f_z) = T(f), \quad P(z) = Q(z) = 1, \quad f_z = f, \quad g_z = g.$$

Putting these together with (1.1), and by some elementary computations on on the indices $q_i, p_i, \theta, p', q', p, q$, we find

$$\left|\int T(f)g\right| \le A_0^{1-\theta} A_1^{\theta} \|f\|_p \|g\|_{q'}.$$

□

2 Fourier series

Having covered the preliminary work, we are now ready to start with the proper contents of the course. To begin with, we will work in the setting of ***periodic*** functions. These are functions defined on the torus $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$.

Our main question in this chapter will be whether one can represent the functions in $C^0(\mathbb{S}^1)$ in terms of linear combinations of the atoms $\{e^{2\pi inx} : n \in \mathbb{Z}\}$. We can trace the origin of this problems to Joseph Fourier's work on the ***heat equation***.

We may begin by trying to find a ***trigonometric polynomial*** on the functions $e^{2\pi inx}$ of the form

$$P(x) = \sum_{|n|\leq N} a_n e^{2\pi inx}$$

that approximates an arbitrary function $f \in C^0(\mathbb{S}^1)$, as N approaches infinity. This is partly motivated by the fact that these functions satisfy an orthogonality property:

$$\int_0^1 e^{2\pi i(n-m)x}\,\mathrm{d}x = \delta_{n,m}.$$

Hence, we may extract the coefficients a_n from P via the identity

$$a_n = \int_0^1 P(x)e^{-2\pi inx}\,\mathrm{d}x.$$

For a general function $f \in C^0(\mathbb{S}^1)$, this expression defines a complex number, and we may denote it by

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,\mathrm{d}x.$$

Notice that if $P(x)$ is a trigonometric polynomial, then we have

$$P(x) = \sum_{n\in\mathbb{Z}} \widehat{P}(m)e^{2\pi imx}, \tag{2.1}$$

and in hopes of generalizing this, we can define (at least formally) the ***Fourier series*** of any $f \in C^0(\mathbb{S}^1)$,

$$Sf(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inx}.$$

Now the ***key question*** that naturally arises is:

Can (2.1) *be generalized to functions $f \in C^0(\mathbb{S}^1)$? That is, does Sf converge?, and is $Sf(x) = f(x)$, at least in some suitable sense?*

The first answer to this problem was only obtained in the early half of the 20th century. As it turns out, this is not the most general setting in which the question can be positively answered. Instead, we may look at $L^1(\mathbb{S}^1)$.

Definition 2.1. Assume that $f \in L^1(\mathbb{S}^1)$. We denote its n-th Fourier coefficient by $\widehat{f}(n)$, and it is defined via

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,\mathrm{d}x.$$

Notice that for $1 \le p \le \infty$, the embedding $L^p(\mathbb{S}^1) \hookrightarrow L^1(\mathbb{S}^1)$ given by the Hölder inequality provides an extension of this definition to any function $f \in L^p(\mathbb{S}^1)$. Indeed,

$$\|f\|_{L^1(\mathbb{S}^1)} \le \|f\|_{L^p(\mathbb{S}^1)} \left\|\mathbb{1}_{[0,1]}\right\|_{L^q(\mathbb{S}^1)} = \|f\|_{L^p(\mathbb{S}^1)}\,.$$

Moreover, notice that through this embedding we can define $\widehat{f}(n)$ for any $f \in L^p(\mathbb{S}^1)$ and any $1 \le p \le \infty$.

2.1 Fourier series of continuous functions

Let us focus first on the Fourier representation of $C^0(\mathbb{S}^1)$ and $C^\alpha(\mathbb{S}^1)$ functions and the properties concerning it.

Definition 2.2. Given $f \in C^s(\mathbb{S}^1)$, $s = 0, 1$, for $N \ge 0$ we define the truncated Fourier series

$$S_N f(x) = \sum_{|n|\le N} \widehat{f}(n)e^{2\pi inx}.$$

Let us consider the most naive setting, and the strongest one. That is, we ask ourselves whether or not there is convergence in the pointwise, uniform sense, of $S_N f$ to f as $N \to \infty$. Notice that by simply working with the definitions, we find

$$S_N f(x) = \int_0^1 \sum_{|n|\le N} e^{-2\pi iny}e^{2\pi inx} f(y)\,\mathrm{d}y = \int_0^1 D_N(x-y)f(y)\,\mathrm{d}y = D_N * f(x), \tag{2.2}$$

where $D_N(x) := \sum_{|n|\le N} e^{2\pi inx}$ is the ***Dirichlet kernel***. A first observation that follows from this is the normalization

$$\int_0^1 D_N(x)\,\mathrm{d}x = 1,$$

but in order to understand (2.2), this is not enough. We may therefore try to extract an explicit expression for D_N. Indeed, by running the computations, we find

$$\begin{aligned} D_N(x) &= e^{-2\pi iNx}\left(1 + e^{2\pi ix} + \dots + e^{2\pi i2Nx}\right) \\ &= e^{-2\pi iNx}\frac{e^{2\pi i(2N+1)x} - 1}{e^{2\pi ix} - 1} \cdot \frac{e^{-\pi ix}}{e^{-\pi ix}} = \frac{\sin[\pi(2N+1)x]}{\sin(\pi x)}. \end{aligned}$$

An elementary analysis of this closed expression reveals that D_N has a maximum at $x = 0$ which grows unboundedly as N approaches infinity, and it shows an oscillatory behavior away from the origin. Since by (2.2), $S_N f(x) = D_N * f(x)$, we may reduce the question of convergence further

$$D_N * f(x) - f(x) = \int_0^1 D_N(y) f(x-y)\,\mathrm{d}y - f(y) = \int_0^1 D_N(x-y)[f(x-y) - f(x)]\,\mathrm{d}y$$

using the normalization property of D_N. In light of this reduction, we come up with the following strategy to show that this converges to zero as $N \to \infty$.

Since D_N becomes singular at the origin, but away from it, this is a highly oscillatory function, we should expect that it integrates zero away from $x = 0$ due to cancellation effects. If we integrate the numerator $\sin[(2N+1)\pi x]$ against a highly regular function (say $f \in C^\infty(\mathbb{S}^1)$), then

$$\int_0^1 f(x) \sin[(2N+1)\pi x]\,\mathrm{d}x = \frac{1}{\pi(2N+1)} \int_0^1 \cos((2N+1)\pi x) f'(x)\,\mathrm{d}x = \dots$$

and we may extract a bound

$$\left| \int_0^1 f(x) \sin[(2N+1)\pi x]\,\mathrm{d}x \right| \le \frac{C_M(f)}{N^M}.$$

The issue we encounter at this point is that we may not acquire any information on the smoothness of $f(y)/\sin \pi y$. Nonetheless, the intuition behind the argument we just exposed can still be shown to work when a Hölder continuity property is appropriately exploited. As a matter of fact, we may exhibit this convergence in a ***quantitative*** sense.

Definition 2.3. For $0 < \alpha < 1$, we define the space of α-Hölder continuous functions $C^\alpha(\mathbb{S}^1) \subset C^0(\mathbb{S}^1)$ as the set of functions f on $\mathbb{S}^1$ such that

$$[f]_{C^\alpha} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

Notice that if $f \in C^1(\mathbb{S}^1)$, then $[f]_{C^\alpha} \le \|f'\|_{L^\infty}$. Moreover, we can take f into account in $[f]_{C^\alpha}$ in a more explicit way by setting

$$\|f\|_{C^\alpha} \coloneqq \|f\|_{L^\infty} + [f]_{C^\alpha}.$$

Theorem 2.4. Assume that $f \in C^\alpha(\mathbb{S}^1)$ for some $0 < \alpha < 1$. Then $S_N f \to f$ uniformly pointwise as $N \to \infty$. Moreover, there is $\gamma = \gamma(\alpha)$ and $C = C(\alpha)$ such that

$$\|S_N f - f\|_{L^\infty} \le C N^{-\gamma} \|f\|_{C^\alpha}.$$

Proof. We use the normalization property of D_N and the periodicity of f to compute

$$\begin{aligned} S_N f(x) - f(x) &= \int_0^1 D_N(x-y) f(x)\,\mathrm{d}y \\ &= \int_0^1 D_N(y)[f(x-y) - f(x)]\,\mathrm{d}y = \int_{-1/2}^{1/2} D_N(y)[f(x-y) - f(x)]\,\mathrm{d}y. \end{aligned}$$

At this point we may divide the integral into two different regions, one containing the origin (and the singularity) and the other one over the complement,

$$A = \int_{|y|\leq\delta} D_N(y)[f(x-y)-f(x)]\,\mathrm{d}y,$$
$$B = \int_{1/2\geq|y|>\delta} D_N(y)[f(x-y)-f(x)]\,\mathrm{d}y.$$

The first integral can be bounded by exploiting the Hölder continuity of f, since we have an effective bound for $f(x-y)-f(x)$. Indeed, $|y|^\alpha$ suffices to counteract the singularity of D_N at the origin,

$$|A| \leq C\int_{|y|\leq\delta} \frac{1}{|y|}|y|^\alpha\,\mathrm{d}y = \tilde{C}_\alpha\delta^\alpha.$$

For the second integral, we intend to perform, essentially, integration by parts, while avoiding the lack of regularity of f. To this end, we compute

$$B = \int_{1/2\geq|y|>\delta} D_N(y)[f(x-y)-f(x)]\,\mathrm{d}y = \int_{1/2\geq|y|>\delta} \sin[\pi(2N+1)y]\frac{f(x-y)-f(x)}{\sin(\pi y)}\,\mathrm{d}y.$$

The oscillatory character of the sin yields $\sin(\pi(2N+1)y) = -\sin\Big(\pi(2N+1)\Big(y+\frac{1}{2N+1}\Big)\Big)$. Renaming

$$h_x(y) = \frac{f(x-y)-f(x)}{\sin(\pi y)}$$

and plugging this into the above,

$$\begin{aligned}
B &= -\int_{1/2\geq|y|>\delta} \sin\left(\pi(2N+1)\left(y+\frac{1}{2N+1}\right)\right)h_x(y)\,\mathrm{d}y\\
&= -\int_{1/2\geq\left|z-\frac{1}{2N+1}\right|>\delta} \sin[(2N+1)\pi z]h_x\left(z-\frac{1}{2N+1}\right)\mathrm{d}z\\
&= -\int_{1/2\geq|z|>\delta} \sin[(2N+1)\pi z]h_x\left(z-\frac{1}{2N+1}\right)\mathrm{d}z\\
&\qquad + \int_{[\delta,\delta+\frac{1}{2N+1}]} \sin[(2N+1)\pi z]h_x\left(z-\frac{1}{2N+1}\right)\mathrm{d}z\\
&\qquad - \int_{[-\delta,-\delta+\frac{1}{2N+1}]} \sin[(2N+1)\pi z]h_x\left(z-\frac{1}{2N+1}\right)\mathrm{d}z.
\end{aligned}$$

At this point, we infer that

$$\begin{aligned}
2B = &-\int_{1/2\geq|z|>\delta} \sin[(2N+1)\pi z]\left[h_x(z)-h_x\left(z-\frac{1}{2N+1}\right)\right]\mathrm{d}z\\
&\qquad + \int_{[\delta,\delta+\frac{1}{2N+1}]} \sin[(2N+1)\pi z]h_x\left(z-\frac{1}{2N+1}\right)\mathrm{d}z\\
&\qquad - \int_{[-\delta,-\delta+\frac{1}{2N+1}]} \sin[(2N+1)\pi z]h_x\left(z-\frac{1}{2N+1}\right)\mathrm{d}z.
\end{aligned}$$

We can immediately estimate the last two integrals by assuming that $\frac{1}{N} \ll \delta$. Indeed, by picking $\delta = N^{-\gamma}$ with $0<\gamma<1$, we will satisfy this condition. For the second integral, we bound

$$\left|h_x\left(z-\frac{1}{2N+1}\right)\right| = \left|\frac{f\left(x-z+\frac{1}{2N+1}\right)-f(x)}{\sin\left[\pi\left(z-\frac{1}{2N+1}\right)\right]}\right| \leq \frac{2\,\|f\|_{L^\infty}}{D\delta} = E\frac{\|f\|_{L^\infty}}{\delta},$$

and since $\sin((2N+1)\pi z) \leq 1$, we reach the estimate

$$\left|\int_{[\delta,\delta+\frac{1}{2N+1}]} \sin[(2N+1)\pi z] h_x\left(z-\frac{1}{2N+1}\right) \mathrm{d}z\right| \leq \frac{E\,\|f\|_{L^\infty}}{(2N+1)\delta}.$$

The same argument works to bound the last integral in the expression of B, and we are left to control the first one, which we begin to bound by noticing that

$$\begin{aligned}\left|h_x(z)-h_x\left(z-\frac{1}{2N+1}\right)\right| &\leq \left|\frac{f(x-z)-f\left(x-z+\frac{1}{2N+1}\right)}{\sin(\pi z)}\right| \\ &\quad + \left|f\left(x-z+\frac{1}{2N+1}\right)+f(x)\right|\left|\frac{1}{\sin(\pi z)}-\frac{1}{\sin\left(\pi\left(z-\frac{1}{2N+1}\right)\right)}\right| \\ &\leq C_1\frac{1}{N^\alpha}\delta^{-1}[f]_{C^\alpha} + C_2\,\|f\|_{L^\infty}\,\frac{1}{N}\delta^{-2}.\end{aligned}$$

Hence we follow with the estimate

$$|B| \leq C_\alpha\left(N^{-1}\delta^{-1}+N^{-\alpha}\delta^{-1}+N^{-1}\delta^{-2}\right)\|f\|_{C^\alpha}.$$

Returning to A, for which we found $|A| \leq C_\alpha \delta^\alpha \|f\|_{C^\alpha}$, we conclude that

$$|D_N * f(x) - f(x)| \leq C_\alpha \|f\|_{C^\alpha}\left(\delta^\alpha + N^{-1}\delta^{-1} + N^{-\alpha}\delta^{-1} + N^{-1}\delta^{-2}\right).$$

At this point we may optimize the choice of $\delta = N^{-\gamma}$ for $\gamma \in (0,1)$ in order to achieve the fastest possible convergence. □

Remark. Back in the proof we exploited the two different expressions of B in order to introduce the Hölder regularity of f after a change of variables. In paractice, this is a feature of the oscillatory behavior of trigonometric functions, and we can take advantage of it in order to show decay of the Fourier coefficients of a function, since $e^{i\pi} = -1$, together with the continuity of translations in the L^1-norm.

In the proof, we do exactly this, but exploit the quantitative vanishing bound that the Hölder regularity of f provides instead of the continuity of translations. It also becomes clear that assuming continuity of f simply yields a non-quantitative bound on B, but it does not suffice to control the singularity at the origin.

When asking oneself whether or not this convergence result can hold similarly under no conditions on the regularity of f (other than continuity), it becomes a much more difficult question that we will address later on. However, a first way to rephrase this question would be as follows.

Assuming that $\{c_n\}_{n\in\mathbb{Z}}$ are the Fourier coefficients of a function $f \in C^0(\mathbb{S}^1)$, can the function f be recovered in a pointwise sense?

This was shown to be the case by L. Fejér, only he showed it is not enough to look at $S_N f$. The main idea behind the proof provided was to consider averages of $S_N f$.

Definition 2.5. We define the Cesàro means of f, $\sigma_N f$, via

$$\sigma_N f(x) = \frac{1}{N}\sum_{n=0}^{N-1} S_n f(x).$$

Explicitly, we have

$$\sigma_N f(x) = \int_0^1 K_N(x-y)f(y)\,\mathrm{d}y = K_N * f(x),$$

where $K_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)$ is the Fejér kernel, which can furthermore be expressed as

$$K_N(x) = \frac{1}{N}\sum_{|n|\le N}(N-|n|)e^{2\pi i n x} = \frac{1}{N}\left(\sum_{|n|\le N-1} e^{-2\pi i\frac{N-n}{2}}\right)^2.$$

Remarkably, this can be shown to be the square of a real number by means of a geometric series representation and a few trigonometric identities,

$$K_N(x) = \frac{1}{N}\left(\frac{e^{\pi i N x}-e^{-\pi i N x}}{e^{\pi i x}-e^{-\pi i x}}\right)^2 = \frac{1}{N}\left(\frac{\sin(\pi N x)}{\sin(\pi x)}\right)^2. \tag{2.3}$$

This reveals explicitly the nonnegative nature of $K_N(x)$ and it is a particular case of an approximation of the identity.

Definition 2.6. We say that a family $\{\varphi_n\}_{n\ge1}$ is an approximation of the identity -or an approximate identity- on $\mathbb{S}^1$ provided that it satisfies the following properties.

(i) ***Normalization:*** $\int_0^1 \varphi_n(x)\,\mathrm{d}x = 1$.

(ii) ***Uniform boundedness:*** $\sup_{N\ge1}\|\varphi_N\|_{L^1(\mathbb{S}^1)} < \infty$.

(iii) ***Concentration:*** $\forall\delta>0$,

$$\lim_{n\to\infty}\int_{-1/2}^{1/2}|\varphi_n(x)|\,\mathbb{1}_{\{|x|>\delta\}}\,\mathrm{d}x = 0.$$

The functions K_N are immediately shown to satisfy these properties, and the answer to the aforementioned question actually follows from a more general result.

Theorem 2.7. Let $\{\varphi_N\}_{N\ge1}$ be an approximate identity, and $f\in C^0(\mathbb{S}^1)$. Then $\lim_{N\to\infty}\varphi_N * f(x) = f(x)$ uniformly in $x\in\mathbb{S}^1$.

Proof. We compute

$$\varphi_N * f(x) - f(x) = \int_{-1/2}^{1/2}\varphi_N(x-y)f(y)\,\mathrm{d}y - f(x) = \int_{-1/2}^{1/2}\varphi_N(x-y)[f(y)-f(x)]\,\mathrm{d}y. \tag{2.4}$$

Given $\varepsilon>0$ and letting $M := \sup_N\|\varphi_N\|_{L^1(\mathbb{S}^1)}$, we may choose $\delta>0$ such that

$$|f(x)-f(y)| < \frac{\varepsilon}{2M}$$

for each x, y with $|x-y| < \delta$, and in view of the definition of an approximation of the identity, and split the previous integral into two different regions: one where we may exploit this condition, and another one where we may exploit the convergence property of φ_N outside $|x-y| < \delta$ to zero. Indeed, (2.4) reads

$$\varphi_N * f(x) - f(x) = A_N + B_N = \int_{-1/2}^{1/2} \varphi_N(x-y)[f(y)-f(x)]\mathbb{1}_{\{|x-y|<\delta\}}\,\mathrm{d}y + \int_{-1/2}^{1/2} \varphi_N(x-y)[f(y)-f(x)]\mathbb{1}_{\{|x-y|\geq\delta\}}\,\mathrm{d}y.$$

Then,

$$|A_N| \leq \|\varphi_N\|_{L^1(\mathbb{S}^1)} \left\| [f(x)-f(y)]\mathbb{1}_{\{|x-y|<\delta\}} \right\|_{L^\infty} \leq M\frac{\varepsilon}{2M} = \frac{\varepsilon}{2},$$

through Hölder's inequality, and if N is large enough,

$$|B_N| \leq \left\| \varphi_N(x-y)\mathbb{1}_{\{|x-y|\geq\delta\}} \right\|_{L^1(\mathbb{S}^1)} \|f(y)-f(x)\|_{L^\infty} \leq \frac{\varepsilon}{2(1+\|f\|_{L^\infty})} 2\|f\|_{L^\infty} \leq \varepsilon. \qquad \square$$

A particularly interesting consequence of this result is the density properties of trigonometric polynomials in $C^0(\mathbb{S}^1)$ with respect to the L^∞ norm. Using Hölder's inequality, we find that this holds true with respect to the L^p norm for any $1 \leq p \leq \infty$.

2.2 Uniqueness of Fourier representation

Convergence of $S_N f$ to f a.e. if $f \in C^0(\mathbb{S}^1)$ was only settled in 1966, and it is currently still a deep, active research topic. One can, instead, relax the concept of convergence from uniform pointwise convergence to just pointwise convergence,leading to the following question.

Consider a sequence of trigonometric polynomials $\left\{ \sum_{|n|\leq N} c_n e^{2\pi i n x} \right\}_{N\geq 1}$*, and assume that*

$$\lim_{N\to\infty} \sum_{|n|\leq N} c_n e^{2\pi i n x} = f(x), \quad \forall x \in \mathbb{S}^1. \tag{2.5}$$

Is then $c_n = \widehat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x}\,\mathrm{d}x$ *necessarily? That is, are the Fourier coefficients the only way to encode the information of* f *into a series of trigonometric polynomials?*

This is, in fact, the case for functions that are Hölder continuous, but the answer is not so direct when removing the regularity assumption. The underlying issue turns out to be the difference between pointwise convergence and uniform convergence. In fact, if the limit in (2.5) is uniform in $x \in \mathbb{S}^1$, then we do obtain uniqueness, but this argument no longer holds when removing the uniform convergence assumption. This gives rise to a different question.

If $\lim_{N\to\infty} \sum_{|n|\leq N} c_n e^{2\pi i n x} = 0$ *for each* $x \in \mathbb{S}^1$ *in a pointwise, non-uniform sense, can we deduce that* $c_n = 0$ *for all* n*?*

If we weaken the limit assumption to only hold on $x \in \mathbb{S}^1 \setminus E$, where E may be a measure zero set, then in general, it is wrong. This question was shown to be linked to number theory, and in particular to Pisot-Vijayaraghavan numbers in the 1990's. As far as our question is concerned, Georg Cantor provided a positive answer.

Theorem 2.8 (*G. Cantor, 1870*). If $\lim_{N\to\infty}\sum_{|n|\le N} c_n e^{2\pi i n x} = 0$ for each $x \in \mathbb{S}^1$, then $c_n = 0$ for all n.

The strategy is as follows:

(i) We first show that the c_n are uniformly bounded, and effectively, they vanish asymptotically, $\lim_{n\to\infty}|c_n| = 0$.

(ii) In order to force uniformity, we introduce an auxiliary function

$$F(x) = c_0\frac{x^2}{2} + \sum_{n\neq 0} c_n \frac{e^{inx}}{(2\pi i n)^2}$$

that converges uniformly in $\mathbb{S}^1$ thanks to (i), and hence $F \in C^0(\mathbb{S}^1)$. Formally, $F''(x)$ should recover the original series, and by the assumption, F should be linear. Through the derivative

$$DF(x) = \lim_{h\to 0} D_h F(x) = \lim_{h\to 0}\frac{F(x+h) - F(x-h) - 2F(x)}{h^2}, \quad h > 0,$$

we may justify this formal step. Furthermore if $F \in C^2(\mathbb{S}^1)$ then $DF(x)$ agrees with $F''(x)$.

(iii) We show that $DF(x)$ exists for all $x \in \mathbb{R}$ and $DF(x) = 0$.

(iv) If $F \in C^0(\mathbb{R})$ satisfies $DF = 0$, then F is necessarily linear.

(v) If $DF = 0$, $F \in C^0(\mathbb{R})$ and $F(x) = c_0\frac{x^2}{2} + \sum_{n\neq 0}\frac{e^{2\pi i n x}}{(2\pi i n)^2}c_n$, then $c_n = 0$.

We will now break these steps down into a collection of lemmas.

Lemma 2.9. Assume that $\{c_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ is a sequence such that

$$\lim_{n\to\infty}\sum_{|n|\le N} c_n e^{2\pi i n x} = 0, \quad \forall x \in \mathbb{S}^1.$$

Then $\lim_{n\to\infty}|c_n| = 0$, and in particular, the sequence $\{c_n\}$ is bounded.

Proof. We replace $2\pi n$ by n in order to simplify the notation, corresponding to the identification $\mathbb{S}^1 \equiv [0, 2\pi]/(0 \sim 2\pi)$. We also pass to real functions from complex functions, i.e.

$$\begin{aligned} c_n e^{inx} + (c_{-n})e^{-inx} &= \sum_{\pm}(\operatorname{Re} c_{\pm n} + i \operatorname{Im} c_{\pm n})(\cos(\pm nx) + i\sin(\pm nx)) \\ &= (a_n\cos(nx) + b_n\sin(nx)) + i(a_n'\cos(nx) + b_n'\sin(nx)), \end{aligned}$$

with the coefficients satisfying

$$a_n^2 + b_n^2 + (a_n')^2 + (b_n')^2 = 2\left(|c_n|^2 + |c_{-n}|^2\right).$$

If $\lim_{n\to\infty} \sum_{|n|\le N} c_n e^{2\pi i n x} = 0$ for all x, then

$$\lim_{N\to\infty}\left[\sum_{|n|\le N+1} c_n e^{inx} - \sum_{|n|\le N} c_n e^{inx}\right] = 0,$$

meaning that, in terms of our coefficients,

$$\lim_{n\to\infty}(a_n \cos nx + b_n \sin nx) = \lim_{n\to\infty}(a_n' \cos nx + b_n' \sin nx) = 0, \quad \forall x \in \mathbb{S}^1.$$

In light of this condition, it suffices to show that $|a_n| + |b_n| \to 0$ as $n \to \infty$. We may express

$$a_n \cos nx + b_n \sin nx = \sqrt{a_n^2 + b_n^2}\cos(nx - \theta_n)$$

where $\theta_n \in [0, 2\pi]$ is chosen -through a trigonometric identity- so that

$$\cos\theta_n = \frac{a_n}{\sqrt{a_n^2+b_n^2}}, \quad \sin\theta_n = \frac{b_n}{\sqrt{a_n^2+b_n^2}}.$$

We carry out the proof by contradiction, so we may assume that $|a_n| + |b_n| \to 0$ fails to take place, and there is a subsequence n_k of indices such that $a_n^2 + b_n^2$ does not converge to 0 as $k \to \infty$. Then, we construct $x_\star \in \mathbb{S}^1$ such that the convergence

$$\sqrt{a_{n_k}^2 + b_{n_k}^2}\cos(n_k x_\star - \theta_{n_k}) \underset{k\to\infty}{\longrightarrow} 0$$

does not take place. In order to do this, we first assume that $n_{k+1} > 100 n_k$, possibly, since passing to subsequences preserves the condition that $\liminf_{k\to\infty} a_{n_k}^2 + b_{n_k}^2 > 0$. Then, we choose

$$x_\star = \sum_{k\in\mathbb{N}} \frac{\alpha_k}{n_k},$$

where $\alpha_n \in [-\pi, \pi]$ and are chose inductively via the requirement that

$$n_k\left(\sum_{1\le l<k}\frac{\alpha_k}{n_k}\right) - \theta_{n_k} + \alpha_k \in [-\pi/4, \pi/4] \mod 2\pi,$$

so that $\cos(n_k\theta_{n_k} - \alpha_k)$ is positive, but very small. Now, notice that

$$n_k x_\star - \theta_{n_k} = n_k\left(\sum_{1\le l<k}\frac{\alpha_l}{n_l}\right) + \alpha_k - \theta_{n_k} + n_k\sum_{l>k}\frac{\alpha_l}{n_l},$$

and since the tail-end of the series converges rapidly to zero, the entire sum is located in $[-\pi/3, \pi/3] \mod 2\pi$. Hence $|\cos(n_k x_\star - \theta_{n_k})| > 0$, and we reach a contradiction since in particular $\sqrt{a_{n_k}^2 + b_{n_k}^2}\cos(n_k x_\star - \theta_{n_k})$ does not converge to zero. □

Hence the series

$$F(x) = c_0\frac{x^2}{2} + \sum_{n\ne 0} c_n \frac{e^{inx}}{(in)^2}$$

converges absolutely and defines a continuous function, since it is the L^∞ limit (the uniform limit) of continuous functions.

Lemma 2.10. Under the assumption that $\sum_{n\in\mathbb{Z}} c_n e^{inx} = 0$ pointwise, $DF(x)$ exists for all $x \in \mathbb{R}$, and vanishes identically.

Proof. Using the definitions,

$$D_h F(x) = c_0 \frac{D_h(x^2)}{2} + \sum_{n\neq 0} c_n \frac{e^{in(x+h)} + e^{in(x-h)} - 2e^{inx}}{(in)^2 h^2}.$$

Note that

$$\begin{aligned}\frac{e^{in(x+h)} + e^{in(x-h)} - 2e^{inx}}{h^2} &= e^{inx}\frac{e^{inh} + e^{-inh} - 2}{h^2} \\ &= e^{inx}\frac{(e^{inh/2} - e^{-inh/2})^2}{h^2} = e^{inx}\frac{1}{-i\frac{h}{2}^2}\sin\left(\frac{nh}{2}\right).\end{aligned}$$

By plugging this in the expression of $D_h F$, we find that

$$D_h F(x) = c_0 + \sum_{n\neq 0} c_n e^{inx}\left[\frac{1}{nh/2}\sin\left(\frac{nh}{2}\right)\right]^2.$$

Exploiting the hypothesis and subtracting, this is

$$D_h F(x) = \sum_{n\neq 0} c_n e^{inx}\left[\left(\frac{1}{nh/2}\sin\left(\frac{nh}{2}\right)\right)^2 - 1\right].$$

At this point, the structure of the above sum reveals that, as $h \to 0$, this will likely vanish. However, we need to invoke ***summation by parts***. Write $a_0 = c_0$ and $a_n = c_n e^{inx} + c_{-n}e^{-inx}$ for $n \geq 1$. We rewrite the sum as

$$S = \sum_{n\geq 1} a_n \left[\left(\frac{1}{nh/2}\sin\left(\frac{nh}{2}\right)\right)^2 - 1\right].$$

For $n \geq 0$, we introduce $s_n := \sum_{k=0}^{n} a_k$. Then for $n \geq 1$, we have $a_n = s_n - s_{n-1}$. This way,

$$s_n = c_0 + \sum_{|k|\leq n} c_k e^{ikx} \underset{n\to\infty}{\longrightarrow} 0.$$

At this point we can identify ***two competing sources of smallness***. In order to exploit this behavior, we split S into two different sums,

$$\begin{aligned}S = &\sum_{1\leq n\leq N}(s_n - s_{n-1})\left[\left(\frac{1}{nh/2}\sin\left(\frac{nh}{2}\right)\right)^2 - 1\right] \\ &+ \sum_{n>N}(s_n - s_{n-1})\left[\left(\frac{1}{nh/2}\sin\left(\frac{nh}{2}\right)\right)^2 - 1\right] = L_N + G_N.\end{aligned}$$

We will use summation by parts on the second sum, moving the difference from the s_n's to the bracketed part by means of a telescopic series. This yields

$$G_N = \sum_{n>N}(s_n - s_{n-1})\left[\left(\frac{1}{nh/2}\sin\left(\frac{nh}{2}\right)\right)^2 - 1\right]$$

$$= \sum_{n>N} s_n \left(\left[\frac{1}{nh/2} \sin\left(\frac{nh}{2}\right)\right]^2 - \left[\frac{1}{(n+1)h/2} \sin\left(\frac{(n+1)h}{2}\right)\right]^2 \right)$$
$$- s_N \left[\left(\frac{1}{(N+1)h/2} \sin \frac{(N+1)h}{2} \right)^2 - 1 \right] = R_N + E_N.$$

Then $S = L_N + R_N + E_N$. Let us denote

$$\Delta_{n,h} := \left[\left[\frac{1}{nh/2} \sin\left(\frac{nh}{2}\right)\right]^2 - \frac{1}{(n+1)h/2} \sin\left(\frac{(n+1)h}{2}\right)\right]^2 = f\left(\frac{nh}{2}\right) - f\left(\frac{(n+1)h}{2}\right),$$

where $f(x) = \frac{\sin^2 x}{x}$ is a $C^\infty(\mathbb{R})$ function. By the Fundamental Theorem of Calculus,

$$\Delta_{n,h} = \int_{\frac{nh}{2}}^{\frac{(n+1)h}{2}} f'(x)\,\mathrm{d}x,$$

hence we may bound

$$\sum_{n>N} |\Delta_{n,h}| \le \sum_{n>N} \int_{\frac{nh}{2}}^{\frac{(n+1)h}{2}} |f'(x)|\,\mathrm{d}x \le \|f'\|_{L^1(\mathbb{R})} < \infty,$$

uniformly in N, h.

We now conclude the proof of the lemma as follows. Given $\varepsilon > 0$, pick $N \in \mathbb{N}$ large enough such that

$$|s_n| < \frac{\varepsilon}{2\,\|f'\|_{L^1(\mathbb{R})}}, \quad \text{for } n > N.$$

Then

$$R_N = \sum_{n>N} s_n \left(\left[\frac{1}{nh/2} \sin\left(\frac{nh}{2}\right)\right]^2 - \left[\frac{1}{(n+1)h/2} \sin\left(\frac{(n+1)h}{2}\right)\right]^2 \right)$$
$$\le \sum_{n>N} |s_n|\,|\Delta_{n,h}| \le \sup_{n>N} |s_n| \sum_{n>N} |\Delta_{n,h}| < \frac{\varepsilon}{2\,\|f'\|_{L^1(\mathbb{R})}} \|f'\|_{L^1(\mathbb{R})} = \frac{\varepsilon}{2},$$

uniformly in h, which provides control on the tail-end of the series. For the initial terms, plus the remaining N-th, we can estimate

$$L_N + E_N \le \left| \sum_{1\le n\le N} (s_n - s_{n-1}) \left[\left(\frac{1}{nh/2} \sin\left(\frac{nh}{2}\right)\right)^2 - 1 \right] \right| + \left| s_N \left[\left(\frac{\sin\left(\frac{(N+1)h}{2}\right)}{(N+1)h/2} \right)^2 - 1 \right] \right|$$
$$\le N \sup_{n\ge 0} |s_n| \max_{1\le n\le N} \left| \left(\frac{\sin\left(\frac{nh}{2}\right)}{nh/2} \right)^2 - 1 \right|.$$

Given fixed N, we may choose h small enough so that this is smaller than $\varepsilon/2$. We conclude by first choosing N so that R_N is controlled by $\varepsilon/2$, and then letting h small enough such that $L_N + E_N$ is controlled by $\varepsilon/2$ as well. Hence $\lim_{h\to 0} D_h F(x) = 0$ for all $x \in \mathbb{R}$. □

We will conclude using the following lemma.

Lemma 2.11. Let G be a continuous function on some interval $I \subset \mathbb{R}$ which satisfies $DG(x) = 0$ for all $x \in \operatorname{Int} I$. Then G is a linear function.

Proof. We work on an interval $I = (a, b)$. The idea is to invoke a suitable perturbation argument to show that G has to be both convex and concave at the same time, in a sandwiching fashion, after a limiting procedure.

Specifically, we first replace G by $G_\varepsilon = G(x) + \varepsilon x^2$ for some small, positive ε. Then $DG_\varepsilon = DG + 2\varepsilon > 0$ since $DG = 0$. It suffices to observe that if this is the case, then G_ε is convex in $[a, b]$.

To this end, we may assume this is false. Thus, there exist two points $c, d \in [a, b]$ such that the graph of G_ε does not lie underneath the straight line segment connecting $(c, G_\varepsilon(c))$ and $(d, G_\varepsilon(d))$. We may add a suitable linear function to G_ε which does not change DG_ε and ***flattens*** the segment connecting $(c, G_\varepsilon(c))$ and $(d, G_\varepsilon(d))$. Hence we may assume that $G_\varepsilon(c) = G_\varepsilon(d)$. Then G_ε attains a maximum in $[c, d]$ which is strictly larger than $G_\varepsilon(c) = G_{\varepsilon(d)}$. Assume this is achieved at $x_* \in (c, d)$. Then

$$D_h G_\varepsilon = \frac{G_\varepsilon(x_* + h) + G_\varepsilon(x_* - h - 2G_\varepsilon(x_*))}{h} \leq 0$$

under the hypothesis that $x_* \pm h \in (c, d)$, which is true for h small enough. Therefore $DG_\varepsilon(x_*) \leq 0$, contradicting $DG_\varepsilon > 0$.

This shows that G_ε is convex. Setting $G_{-\varepsilon}(x) = G(x) - \varepsilon x^2$, we can show, using the same argument, that $G_{-\varepsilon}$ is concave for all $\varepsilon > 0$. At the limit as ε approaches zero, we find that G is both convex and concave, meaning it must be linear. □

We are ready to conclude the proof of the Theorem.

Proof of Theorem 2.8. By the preceding lemmas, F must be linear, and we may write

$$F(x) = \alpha + \beta x, \quad \alpha, \beta \in \mathbb{R}.$$

Notice that, letting x be large enough,

$$cx^2 < \left| |c_0| \frac{|x|^2}{2} - \left| \sum_{n \neq 0} c_n \frac{e^{inx}}{(in)^2} \right| \right| \leq |F(x)| \leq |\alpha| + |\beta|\,|x| < cx^2,$$

for some $c > 0$, and we conclude that $c_0 = 0$. Moreover, since the term with the series is bounded uniformly in x, we conclude from

$$|F(x)| = |\alpha + \beta x| \leq \sum_{n \neq 0} \frac{|c_n|}{n^2}$$

that $\beta = 0$ as well (again by letting x large enough). Hence

$$F(x) = \alpha = \sum_{n \neq 0} c_n \frac{e^{inx}}{(in)^2},$$

and since the series converges uniformly,

$$\frac{c_n}{(in)^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} \left(\sum_{k \neq 0} c_k \frac{e^{ikx}}{(ik)^2} \right) \mathrm{d}x = \frac{1}{2\pi} \int_0^{2\pi} \alpha e^{-inx}\, \mathrm{d}x = 0, \quad n \neq 0.$$

Hence $c_n = 0$, which concludes the proof. □

3 L^p-theory of Fourier series

We now turn our attention to the question of convergence in the L^p-sense. For now, we will only be able to understand a small part of the theory - namely, the cases $p = 1, 2, \infty$, and an equivalent condition for the boundedness of the Fourier series operators. We will, however, answer the more complicated questions later, when we have seen the Hilbert transform.

Proposition 3.1. Let $\{\varphi_N\}_{N\geq 1}$ be an approximation of the identity on $\mathbb{S}^1$. Then if $f \in L^p(\mathbb{S}^1)$, for $1 \leq p < \infty$, we have

$$\|\varphi_N * f - f\|_{L^p(\mathbb{S}^1)} \underset{N\to\infty}{\longrightarrow} 0.$$

Moreover, we have that $\sigma_N f \to f$ in the L^p-sense as $N \to \infty$, for every $f \in L^p(\mathbb{S}^1)$ and $1 \leq p < \infty$.

Proof. We reduce to the known case of pointwise convergence for continuous functions by exploiting the density of $C^0(\mathbb{S}^1)$ in $L^p(\mathbb{S}^1)$. Indeed, given $\varepsilon > 0$, we may pick $g \in C^0(\mathbb{S}^1)$ with

$$\|g - f\|_{L^p(\mathbb{S}^1)} < \frac{\varepsilon}{2(M+1)},$$

where $M = \sup_N \|\varphi\|_{L^1(\mathbb{S}^1)}$. By the triangle inequality,

$$\begin{aligned}
\|\varphi_N * f - f\|_{L^p(\mathbb{S}^1)} &= \|\varphi_N * (f-g) + \varphi_N * g - (f-g) - g\|_{L^p(\mathbb{S}^1)} \\
&\leq \|\varphi_N * (f-g)\|_{L^p(\mathbb{S}^1)} + \|\varphi_N * g - g\|_{L^p(\mathbb{S}^1)} + \|f-g\|_{L^p(\mathbb{S}^1)} \\
&\leq \|\varphi_N\|_{L^1(\mathbb{S}^1)} \|f-g\|_{L^p(\mathbb{S}^1)} + \|\varphi_N * g - g\|_{L^p(\mathbb{S}^1)} + \frac{\varepsilon}{2(M+1)} \\
&\leq M\frac{\varepsilon}{2(M+1)} + \|\varphi_N * g - g\|_{L^p(\mathbb{S}^1)} + \frac{\varepsilon}{2(M+1)},
\end{aligned}$$

and we may bound the middle term by using Theorem 2.7 and bounding by the supremum inside the integral,

$$\|\varphi_N * g - g\|_{L^p(\mathbb{S}^1)} \leq \|\varphi_N * g - g\|_{L^\infty(\mathbb{S}^1)} \underset{N\to\infty}{\longrightarrow} 0.$$

□

At this point, we may verify some basic properties of trigonometric polynomials, like the fact that they are dense in $L^p(\mathbb{S}^1)$, and the ***Parseval identity***, which in particular provides a positive answer to the boundedness of Fourier series in L^2.

Corollary 3.2. For $1 \le p < \infty$, the family of trigonometric polynomials is dense in $L^p(\mathbb{S}^1)$. In particular, for $p = 2$, denoting

$$\widehat{f}(n) := \int_0^1 f(x)e^{-2\pi i n x}\,\mathrm{d}x,$$

we have, for $g, f \in L^2(\mathbb{S}^1)$, the ***Parseval identity***

$$\int_{\mathbb{S}^1} f(x)\overline{g(x)}\,\mathrm{d}x = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)} \tag{3.1}$$

Proof. The first part follows since $\sigma_N f$ is a trigonometric polynomial. Moreover, the family $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ form an orthogonal set, which is a basis for $L^2(\mathbb{S}^1)$ because of density. Hence (3.1) follows from a direct computation. □

We may ask the question of convergence of $S_N f$ to f in the L^p-sense. Parseval's identity readily provides an answer in the case $p = 2$. Indeed,

$$\|S_N f - f\|^2_{L^2(\mathbb{S}^1)} = \sum_{|n|>N} \left|\widehat{f}(n)\right|^2,$$

which converges to zero as $N \to \infty$ as it conforms the tail end of the series defining $\|f\|_{L^2(\mathbb{S}^1)}$ through (3.1).

But *what is the answer when* $p \neq 2$? This question becomes much more complicated, and it will be the goal of the course to answer it once we have developed the appropriate aspects of the theory of the Hilbert transform. For now we develop some general tools which allow dealing with the endpoint cases $p = 1$ and $p = \infty$ in the setting of $C^0(\mathbb{S}^1)$. To settle these last two cases, we use the following version of the Banach-Steinhaus principle.

Proposition 3.3. Let $1 \le p \le \infty$. Then $S_N f \to f$ in the L^p-sense for $f \in L^p(\mathbb{S}^1)$ as $N \to \infty$ if and only if there is a uniform operator bound for the operator norm

$$\sup_N \|S_N\|_{L^p\to L^p} = \sup_N \sup_{\|f\|_{L^p(\mathbb{S}^1)}\le 1} \|S_N f\|_{L^p(\mathbb{S}^1)} < \infty.$$

For $p = \infty$, convergence of $S_N f$ to f in $L^\infty(\mathbb{S}^1)$ for holds for all $f \in C^0(\mathbb{S}^1)$ if and only if

$$\sup_N \|S_N\|_{L^\infty\to L^\infty} < \infty.$$

Soon we will verify that

$$\sup_N \|S_N\|_{L^1\to L^1} = \sup_N \|S_N\|_{L^\infty\to L^\infty} = +\infty,$$

meaning that there must be $f \in L^1(\mathbb{S}^1)$ and $f \in L^\infty(\mathbb{S}^1)$ for which the series $S_N f$ does not converge to f in the L^1 and the L^∞-sense respectively.

Proof of Proposition 3.3. We will verify the case $1 \le p < \infty$, and leave the case $p = \infty$ as an exercise. Given $f \in L^p(\mathbb{S}^1)$ and $\varepsilon > 0$, pick a function $g \in C^\alpha$ for some $\alpha \in (0, 1)$ (which we may do by density of $C_c^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$) such that

$$\|f-g\|_{L^p(\mathbb{S}^1)} < \frac{\varepsilon}{3(M+1)}, \quad M = \sup_N \|S_N\|_{L^p\to L^p} < \infty.$$

Then we bound

$$\|S_N f - f\|_{L^p(\mathbb{S}^1)} \leq \|S_N g - g\|_{L^p(\mathbb{S}^1)} + \|S_N(f-g)\|_{L^p(\mathbb{S}^1)} + \|f-g\|_{L^p(\mathbb{S}^1)}\,.$$

Now choose N large enough so that

$$\|S_N g - g\|_{L^p(\mathbb{S}^1)} \leq \|S_N g - g\|_{L^\infty(\mathbb{S}^1)} < \frac{\varepsilon}{3}.$$

Then, we obtain

$$\|S_N f - f\|_{L^p(\mathbb{S}^1)} \leq \frac{\varepsilon}{3} + M\frac{\varepsilon}{3(M+1)} + \frac{\varepsilon}{3} \leq \varepsilon.$$

We now prove the remaining direction. Assuming that $S_N f \to f$, argue by contradiction to find $\{N_l\}_{l\geq 1} \subset \mathbb{N}$ with $\|S_{N_l}\|_{L^p\to L^p} \longrightarrow +\infty$ as $l\to\infty$. Up to a subsequence, we can make this divergence satisfy $\|S_{N_l}\|_{L^p\to L^p} > 2^l$, and starting here, we may construct a ***pathological*** function $f \in L^p(\mathbb{S}^1)$ that will yield a contradiction.

Note that $\|S_{N_l}\|_{L^p\to L^p} > 2^l$ is equivalent to finding $f_l \in L^p(\mathbb{S}^1)$ with $\|f_l\|_{L^p(\mathbb{S}^1)} = 1$ and such that $\|S_N f_l\|_{L^p(\mathbb{S}^1)} > 2^l$. Furthermore, we may assume that f_l is a trigonometric polynomial by density, and we will construct f inductively by

$$f(x) := \sum_{l\geq 1} l^{-2} e^{2\pi i M_l x} f_l(x).$$

The numbers M_l will be chosen inductively, and the sum converges in $L^p(\mathbb{S}^1)$ because the l^{-2} factor forces the absolute summability of the series. We now impose the following conditions on M_l.

(i) $M_l - N_l \to \infty$ as $l \to \infty$.

(ii) Setting, for $l \geq 2$,

$$g_l := \sum_{j=1}^{l-1} j^{-2} e^{2\pi M_j x} f_j(x),$$

then $S_{M_l - N_l - 1} g_l = g_l$.

Indeed, recall that $S_{M_l-N_l-1} g_l$ removes all exponentials in the Fourier representation of g_l with exponent $|n| > M_l - N_l - 1$. So by ensuring that $|n + M_j| \leq M_l - N_l - 1$, where n is any frequency in a function f_j, for $1 \leq j \leq l-1$, then we effectively remove all the frequencies in g_l that are higher than $M_l - N_l - 1$.

(iii) $M_l + n \geq 0$ for any frequency n occurring in f_l.

(iv) $S_{M_l+N_l} f = S_{M_l+N_l} g_{l+1}$.

We now ***claim*** that, assuming (i)-(iv), $S_N f$ does not converge to f in the $L^p(\mathbb{S}^1)$-sense. To begin with, by (iv) and (ii),

$$\begin{aligned} S_{M_l+N_l}f - S_{M_l-N_l-1}f &= S_{M_l+N_l}g_{l+1} - S_{M_l-N_l}g_{l+1} \\ &= S_{M_l+N_l}g_l - S_{M_l-N_l-1}g_l + (S_{M_l+N_l} - S_{m_l-N_l-1})\left(l^{-2}e^{2\pi i M_l x}f_l(x)\right) \\ &= l^{-2}e^{2\pi i M_l x}S_{N_l}f_l(x). \end{aligned}$$

Therefore

$$\|S_{N_l+M_l}f - S_{M_l-N_l-1}f\|_{L^p} = l^{-2}\|S_{N_l}f_l\|_{L^p} > 2^l l^{-2},$$

which diverges as $l\to\infty$. On the other hand, since $M_l - N_l \to \infty$, we have

$$\begin{aligned} &\limsup_{t\to\infty}\|S_{N_l+M_l}f - S_{M_l-N_l-1}f\|_{L^p} \\ &\qquad \le \limsup_{l\to\infty}\|S_{N_l+M_l}f - f\|_{L^p} + \limsup_{l\to\infty}\|S_{M_l-N_l-1}f - f\|_{L^p} = 0, \end{aligned}$$

a contradiction. □

We turn to the main consequence of the proposition.

Corollary 3.4. There is $f \in L^1(\mathbb{S}^1)$ whose Fourier series does not converge to f in $L^1(\mathbb{S}^1)$, and moreover, there is $f \in C^0(\mathbb{S}^1)$ for which the same occurs, in the $L^\infty(\mathbb{S}^1)$-sense.

Proof. By Proposition 3.3, it suffices to show that

$$\sup_N \|S_N\|_{L^1\to L^1} = \sup_{\|f\|_{L^1(\mathbb{S}^1)}\le 1}\|S_N f\|_{L^1(\mathbb{S}^1)} = \infty.$$

Consider K_M, the Fejér kernel: since it is an approximate identity,

$$\|S_N(K_M)\|_{L^1(\mathbb{S}^1)} = \|K_M * D_N\|_{L^1(\mathbb{S}^1)} \underset{M\to\infty}{\longrightarrow} \|D_N\|_{L^1(\mathbb{S}^1)} \ge c\log(N).$$

Hence, $\|S_N\|_{L^1\to L^1} \ge c\log(N)$ and the supremum over $N \in \mathbb{N}$ must be infinite. The case $p=\infty$ follows similarly by noting that

$$\sup_{\|f\|_{L^\infty}\le 1}\|D_N * f\|_{L^\infty} \ge \|D_N\|_{L^1(\mathbb{S}^1)}.$$

□

We will later show that, whenever $p \in (0,1)$, convergence of $S_N f$ to f holds for all $f \in L^p(\mathbb{S}^1)$. But at the moment, we will leave the issues of L^p-convergence in favor of different topics in the L^p-theory of Fourier series. In particular, we will discuss certain types of spaces that are closely related to Fourier series, such as Sobolev spaces.

Let us give some motivation first. Assume that $f \in C^1(\mathbb{S}^1)$. Then, integrating by parts,

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x}\,\mathrm{d}x = -\frac{1}{2\pi i n}\int_0^1 e^{-2\pi i n x}f'(x) = -\frac{1}{2\pi i n}\widehat{f'}(n).$$

We then deduce that

$$\left|\widehat{f}(n)\right| \le \frac{\|f'\|_{L^1(\mathbb{S}^1)}}{|n|},$$

which provides a decay estimate for the Fourier coefficients of f. Furthermore, the more regular the function f is, the better the decay estimate: if $f \in C^M(\mathbb{S}^1)$, then

$$\left|\widehat{f}(n)\right| \leq \frac{C}{|n|^M}.$$

An important question we can ask is whether or not this reasoning can be inverted, i.e., if we may find integrability properties of the derivatives of a function by investigating its decay properties. As we will readily see, the first instance of these will be Bernstein's inequality.

3.1 Bernstein's inequality

Assume that f is a trigonometric polynomial on $\mathbb{S}^1$ of degree N,

$$f(x) = \sum_{|n|\leq N} a_n e^{2\pi i n x}.$$

Then, by Parseval's identity,

$$\|f'\|^2_{L^2(\mathbb{S}^1)} = \sum_{|n|\leq N} (2\pi i n \, |a_n|)^2 \leq (2\pi N)^2 \, \|f\|^2_{L^2}.$$

Hence,

$$\|f'\|_{L^2(\mathbb{S}^1)} \leq CN \, \|f\|_{L^2(\mathbb{S}^1)}.$$

We ask the following question:

What if we replace L^2 by L^p, for $p \in [1, \infty]$?

Parseval's identity no longer works, and we lose a factor N in the previous bound,

$$\|f'(x)\|_{L^p(\mathbb{S}^1)} \leq \sum_{|n|\leq N} 2\pi \, |n| \, |a_n| \leq 2\pi \sum_{|n|\leq N} |n| \, \|f\|_{L^p(\mathbb{S}^1)} \leq CN^2 \, \|f\|_{L^p(\mathbb{S}^1)}.$$

Theorem 3.5 (*Bernstein's inequality*). There exists a positive constant $C > 0$ such that for every trigonometric polynomial f of degree N and $1 \leq p \leq \infty$,

$$\|f'\|_{L^p(\mathbb{S}^1)} \leq CN \, \|f\|_{L^p(\mathbb{S}^1)}.$$

Proof. The key idea is to express f in a way that allows the exploitation of $\widehat{f}(n) = 0$ for $|n| > N$. To this end, we introduce a function $V_(x)$ such that

$$\widehat{V_N}(n) = 1, \quad n \in [-N, N]$$

and consider the function $V_N * f = f$ by passing to Fourier coefficients. Hence, we may exploit the expression of f',

$$f'(x) = (V_N * f)'(x) = \int_0^1 V_N'(x-y) f(y) \, \mathrm{d}y.$$

Thus, by Young's convolution inequality

$$\|f'\|_{L^p(\mathbb{S}^1)} \leq \|V_N'\|_{L^1(\mathbb{S}^1)} \, \|f\|_{L^p(\mathbb{S}^1)}.$$

The function V_N will be chosen as a trigonometric polynomial called the ***de la Vallée-Pousin kernel***. It is defined by

$$V_N(x) = (1 + e^{2\pi i N x} + e^{-2\pi i N x}) K_N(x), \quad N \geq 1, \tag{3.2}$$

where K_N is the Féjer Kernel (2.3). We conclude the proof thanks to the lemma below, which establishes that

$$\left\| V_N' \right\|_{L^1(\mathbb{S}^1)} \leq CN. \qquad \square$$

Lemma 3.6. For V_N defined as in (3.2), we have $\widehat{V_N}(n) = 1$ for each $n \in [-N, N]$, and $\operatorname{supp}(\widehat{V_N}) \subset [-2N, 2N]$. Consequently, we have the bounds

$$\|V_N\|_{L^1(\mathbb{S}^1)} \leq C, \; \left\| V_N' \right\|_{L^1(\mathbb{S}^1)} \leq CN.$$

Proof. Pick $n \in [-N, N]$ and compute $\widehat{V_N}(n)$,

$$\widehat{V_N}(n) = \widehat{K_N}(n) + \widehat{e^{2\pi i N x} K_N}(n) + \widehat{e^{-2\pi i N x} K_N}(n) = \widehat{K_N}(n) + \widehat{K_N}(n - N) + \widehat{K_N}(n + N).$$

Since

$$K_N(x) = \sum_{|n|=0}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x},$$

we have $\widehat{V_N}(n) = 1$ by reading off the coefficients in K_N. Hence,

$$\left\| V_N' \right\|_{L^1(\mathbb{S}^1)} \leq CN \|K_N\|_{L^1(\mathbb{S}^1)} + C \left\| K_N' \right\|_{L^1(\mathbb{S}^1)}.$$

The first norm on the right-hand side is equal to 1, whereas the second norm can be estimated via

$$\left| K_N'(x) \right| = \left| \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{N} \left(\frac{\sin(\pi N x)}{\sin(\pi x)} \right)^2 \right| \leq C \left| x \right|^{-2},$$

and the estimate

$$\left| K_N'(x) \right| \leq CN^2,$$

which follows from the fact that K_N is a trigonometric polynomial of degree N, with coefficients that are bounded by 1, and hence K_N' is of degree N, and with coefficients bounded by N. $\square$

3.2 Sobolev spaces

Oftentimes, the notion of classical derivatives is too restrictive to work with. Furthermore, C^k spaces are often not good enough to work with, since they might lack certain desirable properties, or because showing that a certain function lies in one of these spaces turns out to be very hard. This becomes especially relevant in optimization and variational problems.

To bypass these disadvantages but still measure differentiability in some sense, we work with ***Sobolev spaces*** and ***weak derivatives***. In particular, these generalize the classical notion of derivatives and C^k spaces.

Definition 3.7. We define the distributional derivative of a function $f \in L^1_{\text{loc}}(\mathbb{S}^1)$ as the distribution T that acts on a test function $\phi \in C_c^\infty(\mathbb{S}^1)$ via

$$T(\phi) = -\int_0^1 f(x)\phi'(x)\,\mathrm{d}x.$$

Functions in $L^p(\mathbb{S}^1)$ are all automatically equipped with a distributional derivative. Moreover, if the function f has a derivative in the classical sense, then the distribution T will be given by integration against f'. In this sense, distributional derivatives generalize classical derivatives. Furthermore, it may happen that a function realizes the derivative of f in a distributional sense, which leads to ***weak derivatives***.

Definition 3.8. We say that g is the weak derivative of $f \in L^1_{\text{loc}}(\mathbb{S}^1)$ if, for each test function ϕ,

$$\int_0^1 g(x)\phi(x)\,\mathrm{d}x = -\int_0^1 f(x)\phi'(x)\,\mathrm{d}x.$$

Having introduced these notions, we are ready to define ***Sobolev spaces***. For our interests, we will only work with the ***energy spaces***, a particular case of Sobolev spaces that appears often in PDE problems and that enjoys a very clean Fourier representation.

Definition 3.9. We define $H^1(\mathbb{S}^1)$ as the set of functions $f \in L^2(\mathbb{S}^1)$ which admit a weak derivative in $L^2(\mathbb{S}^1)$.

For functions $f \in H^1(\mathbb{S}^1)$ and through the definition of the weak derivative g of f, we deduce that

$$\widehat{g}(n) = 2\pi i n \widehat{f}(n),$$

which by Parseval's formula means that $\{n\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Conversely, $f \in L^2(\mathbb{S}^1)$ lies in $H^1(\mathbb{S}^1)$ if $\{n\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^2$. In fact, if we assume the latter, then for general $\varphi \in C^1(\mathbb{S}^1)$,

$$\int \varphi'(x)f(x)\,\mathrm{d}x = \lim_{N\to\infty}\int \varphi'(x)S_N f(x)\,\mathrm{d}x.$$

But

$$\left\|(S_N f)' - g\right\|_{L^2(\mathbb{S}^1)} \underset{N\to\infty}{\longrightarrow} 0$$

by Parseval, since $\{n\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^2$. Hence

$$\int \varphi'(x)f(x)\,\mathrm{d}x = -\int \varphi(x)g(x)\,\mathrm{d}x.$$

Definition 3.10. Let $s \in \mathbb{R}$. Then we define the space

$$H^s(\mathbb{S}^1) := \left\{ f \in L^2(\mathbb{S}^1) \colon \sum_{n\neq 0} |n|^{2s} \left|\widehat{f}(n)\right|^2 < \infty \right\},$$

endowed with the norm

$$\|f\|^2_{H^1(\mathbb{S}^1)} = \sum_{n\in\mathbb{Z}} (1+|n|^{2s}) \left|\widehat{f}(n)\right|^2.$$

We now link this to classical function spaces.

Definition 3.11 (*Wiener algebra*). We define

$$\mathcal{A}(\mathbb{S}^1) := \left\{ f \in C^0(\mathbb{S}^1) \colon \sum_{k\in\mathbb{Z}} \left|\widehat{f}(k)\right| < +\infty \right\}.$$

Lemma 3.12. $\mathcal{A}(\mathbb{S}^1)$ is an algebra, and if $f, g \in \mathcal{A}(\mathbb{S}^1)$, then

$$\widehat{fg}(n) = (\widehat{f} * \widehat{g})(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(k)\widehat{g}(n-k).$$

Proof. If f is a trigonometric polynomial, then

$$\widehat{fg}(n) = \int_{\mathbb{S}^1} e^{-2\pi i n x} \left(\sum_{|k|\le N} a_k e^{2\pi i k x} \right) g(x)\,\mathrm{d}x = \cdots = \sum_{|k|\le N} \widehat{f}(k)\widehat{g}(n-k).$$

For a general $f \in \mathcal{A}(\mathbb{S}^1)$, we approximate f with a trigonometric polynomial $K_N * f$. Then $\mathcal{F}((K_N * f)g)\,(n)$ converges uniformly in n to $\widehat{fg}(n)$ as $N \to \infty$, and by the preceding,

$$\mathcal{F}((K_N * f)g)\,(n) = \sum_{|k|\le N} \widehat{K_N * f}(k)\widehat{g}(n-k) = \sum_{|k|\le N} \left(1 - \frac{|k|}{N}\right)_+ \widehat{f}(k)\widehat{g}(n-k),$$

which converges to $\widehat{f} * \widehat{g}(n)$ as $N \to \infty$. By Fubini, we may bound the ℓ^1-norm of $\widehat{f} * \widehat{g}$ by the product of $\|\widehat{f}\|_{\ell^1}$ and $\|\widehat{g}\|_{\ell^1}$. □

Proposition 3.13 (*Sobolev embedding*). For $s > 1/2$, $H^s(\mathbb{S}^1) \subset \mathcal{A}(\mathbb{S}^1)$.

Proof. We want to show that $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$ whenever $f \in H^s(\mathbb{S}^1)$, $s > 1/2$. This follows from Cauchy-Schwarz,

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right| &= \sum_{n\in\mathbb{Z}} (1 + |n|^{2s})^{1/2} \widehat{f}(n) \frac{1}{(1 + |n|^{2s})^{1/2}} \\ &\le \left(\sum_{n\in\mathbb{Z}} (1 + |n|^{2s}) \left|\widehat{f}(n)\right|^2 \right)^{1/2} \left(\frac{1}{1 + |n|^{2s}} \right)^{1/2} < C \|f\|_{H^s}. \end{aligned}$$

□

Now we may ask ourselves what other embeddings we might find - provided, possibly, that we ask that f enjoys at least some regularity. In particular,

if $f \in C^\alpha(\mathbb{S}^1)$, which $H^\beta(\mathbb{S}^1)$ does it embed into?

Proposition 3.14. Whenever $\alpha > \beta$, it holds that $C^\alpha(\mathbb{S}^1) \subset H^\beta(\mathbb{S}^1)$. More precisely, there is $C_{\alpha,\beta} > 0$ such that

$$\|f\|_{H^\beta(\mathbb{S}^1)} \le C_{\alpha,\beta} \|f\|_{C^\alpha(\mathbb{S}^1)} = C_{\alpha,\beta} \Big(\|f\|_{L^\infty(\mathbb{S}^1)} + [f]_{C^\alpha} \Big).$$

The proof introduces a compact version of the Littlewood-Paley decomposition, a tool that allows for frequency localization and that is used very frequently in the context of $\mathbb{R}^n$ when working on certain problems in dispersive PDEs, and that is based on a dyadic decomposition in frequency space. We may -naively- try to show this in a direct way, by computing

$$\widehat{f}(n) = \cdots = \frac{1}{2}\int_0^1 e^{-2\pi inx}\left[f(x) - f\left(x - \frac{1}{2n}\right)\right] \mathrm{d}x,$$

which yields the estimate

$$\left|\widehat{f}(n)\right| \leq [f]_{C^\alpha}\frac{1}{2}\left(\frac{1}{2n}\right)^\alpha .$$

This, however, is not enough, and needs to be significantly strengthened. We will achieve this by considering dyadic frequency ranges $2^j \leq |n| \leq 2^{j+1}$, $j \geq -1$. As a matter of fact, we have the following result.

Lemma 3.15. If $f \in C^\alpha(\mathbb{S}^1)$, then there is $C_\alpha > 0$ such that

$$\sum_{2^j \leq |n| \leq 2^{j+1}} |n|^{2\alpha}\left|\widehat{f}(n)\right|^2 \leq C_\alpha [f]^2_{C^\alpha}.$$

This improves the situation greatly with respect to our previous bound in two main ways. The first one is that we are essentially removing a factor 2^j from the right-hand side. The second one is the fact that the sum of squares in the dyadic decomposition already hints at an orthogonality property - so we may apply Parseval's identity.

Proof of Proposition 3.14. We will assume Lemma 3.15. Assume $f \in C^\alpha(\mathbb{S}^1)$, and $\beta < \alpha$. Then

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |n|^{2\beta}\left|\widehat{f}(n)\right|^2 &= \sum_{j\geq -1}\sum_{2^j < |n| \leq 2^{j+1}} |n|^{2\beta}\left|\widehat{f}(n)\right|^2 \\ &\leq \sum_{j\geq -1} 2^{-2j(\alpha-\beta)}\left(\sum_{2^j<|n|\leq 2j+1} |n|^{2\alpha}\left|\widehat{f}(n)\right|^2\right) \\ &\leq C_\alpha [f]^2_{C^\alpha}\sum_{j\geq -1} 2^{-2j(\alpha-\beta)} \leq D_{\alpha,\beta}[f]^2_{C^\alpha}.\end{aligned}$$

Finally,

$$\|f\|^2_{L^2} \leq \int_0^1 (|f(x) - f(0)| + |f(0)|)^2 \,\mathrm{d}x \leq C_\alpha([f]_{C^\alpha} + \|f\|_{L^\infty})^2 = C_\alpha \|f\|^2_{C^\alpha}.$$ □

We are now left to show Lemma 3.15.

Proof of Lemma 3.15. We will employ a similar idea to that of the proof of Bernstein's theorem, except this time, instead of a function with Fourier support on $[-N, N]$, we consider funtions with Fourier support on $[2^j, 2^{j+1}]$. Define

$$S_{2^j<|n|\leq 2^{j+1}} f := \sum_{2^j<|n|\leq 2^{j+1}} \widehat{f}(n)e^{2\pi inx}.$$

We begin by finding a function $\varphi_j(x)$, which we may call the ***Littlewood-Paley multiplier***, such that φ_j is essentially explicit, and $\widehat{\varphi_j}(n) = 1$ whenever $2^j < |n| \le 2^{j+1}$. Then,

$$S_{2^j<|n|\le 2^{j+1}} f = \varphi_j * S_{2^j<|n|\le 2^{j+1}} f,$$

and so

$$\begin{aligned}\sum_{2^j<|n|\le 2^{j+1}} \left|\widehat{f}(n)\right|^2 &= \left\|\varphi_j * S_{2^j<|n|\le 2^{j+1}} f\right\|_{L^2(\mathbb{S}^1)} = \sum_{n\in\mathbb{Z}} |\widehat{\varphi_j}(n)|^2 \left|\mathcal{F}(S_{2^j<|n|\le 2^{j+1}} f)(n)\right|^2 \\ &\le \sum_{n\in\mathbb{Z}} |\widehat{\varphi_j}(n)|^2 \left|\widehat{f}(n)\right|^2 = \|\varphi_j * f\|^2_{L^2(\mathbb{S}^1)}.\end{aligned} \tag{3.3}$$

It remains to bound the norm in the right-hand side by $[f]^2_{C^\alpha}$ and with a $2^{-2j\alpha}$ gain. Before this, we may explicitly construct φ in order to extract the necessary estimates.

Note

To find such a function, recall that

$$V_N(x) = (1 + e^{2\pi i n x} + e^{-2\pi i n x}) K_N(x)$$

has the property that $\widehat{V_N}(n) = 1$ whenever $n \in [-N, N]$, and supp $\widehat{V_N} \subset [-2N, 2N]$. We will choose φ_j as V_N after rescaling and translating appropriately:

$$\varphi_j(x) := \left(e^{2\pi i(3x)2^{j-1}} + e^{-2\pi i(3x)2^{j-1}}\right) V_{2^{j-1}}(x).$$

Since $\widehat{V_{2^{j-1}}} = 1$ if $|n| < 2^{j-1}$, then $\widehat{\varphi_j} = 1$ if $|n| \in [-2^{j+1}, -2^j]$, and it is supported on an annulus. Recalling

$$K_N(x) = \frac{1}{N}\left(\frac{\sin(\pi N x)}{\sin(\pi x)}\right)^2 \le C \min\left\{N, \frac{1}{N|x|^2}\right\},$$

we extract the bound

$$|\varphi_j(x)| \le D \min\left\{2^j, \frac{1}{2^j |x|^2}\right\}.$$

Note now that, working in $[-1/2, 1/2]$, and since $\widehat{\varphi}(0) = 0$,

$$\begin{aligned}\varphi_j * f(x) &= \int_0^1 \varphi_j(y) f(x-y)\,\mathrm{d}y = \int_0^1 \varphi_j(y)\,[f(x-y) - f(x)]\,\mathrm{d}y \\ &= \int_{|y|<2^{-j}} \varphi_j(y)\,[f(x-y) - f(x)]\,\mathrm{d}y + \int_{2^{-j}\le|y|\le 1/2} \varphi_j(y)\,[f(x-y) - f(x)]\,\mathrm{d}y.\end{aligned}$$

We bound the previous two integrals in absolute value. The first one,

$$\left|\int_{|y|<2^{-j}} \varphi_j(y)\,[f(x-y) - f(x)]\,\mathrm{d}y\right| \le \int_{|y|<2^{-j}} 2^j D\,|y|^\alpha\,[f]_\alpha\,\mathrm{d}y \le 2^{1-\alpha j} D [f]_\alpha.$$

The second one,

$$\left|\int_{2^{-j}\le|y|\le 1/2} \varphi_j(y)\,[f(x-y) - f(x)]\,\mathrm{d}y\right| \le \int_{|y|\ge 2^{-j}} 2^{-j} D\,|y|^{\alpha-2}\,[f]_\alpha\,\mathrm{d}y \le \tilde{D} 2^{-\alpha j} [f]_\alpha.$$

We combine the two to obtain $|\varphi_j * f(x)| \leq 2^{-\alpha j} C[f]_\alpha$, and therefore

$$\|\varphi_j * f\|_{L^2(\mathbb{S}^1)}^2 \leq \|\varphi_j * f\|_{L^\infty(\mathbb{S}^1)}^2 \leq 2^{-2\alpha j} C^2 [f]_\alpha^2.$$

Finally, coming back to (3.3),

$$\sum_{2^j < |n| \leq 2^{j+1}} |n|^{2\alpha} \left|\widehat{f}(n)\right|^2 \leq 2^{2\alpha(j+1)} \sum_{2^j < |n| \leq 2^{j+1}} \left|\widehat{f}(n)\right|^2 \leq 2^{2\alpha} C^2 [f]_\alpha^2.$$

□

4 Lacunary Fourier series and Weyl's equidistribution theorem

In this chapter, we explore the relation that Fourier series bear with differentiability, as well as some interactions between number theory and Fourier analysis. In fact, consider the continuous, nowhere differentiable Weierstrass function

$$W(x) = \sum_{n\geq 0} 2^{-n} \cos(2^n \cdot 2\pi x). \tag{4.1}$$

In our framework, Fourier theory can be used to produce infinitely many examples of continuous functions that are nowhere differentiable. We will introduce the concept of Lacunary Fourier series to treat these.

Definition 4.1 (*Lacunary Fourier series*). Assume that $\{\lambda_n\}_{n\geq 0} \subset \mathbb{R}_+$ is a sequence such that there is $a > 1$ with

$$\lambda_{n+1} > a\lambda_n, \quad \forall n \geq 0.$$

Then we say $\{\lambda_n\}_{n\geq 0}$ is lacunary.

The idea in this chapter is that since we may infer some decay properties of the coefficients of the Fourier series of a function f the moment we know about its regularity, we may try to do the opposite now: that is, prescribe the coefficients in a Fourier series, and try to understand the regularity properties (or lack thereof) of the function that their series define.

Theorem 4.2 (*Katznelson*). Assume that $f \in C^0(\mathbb{S}^1)$ given by a lacunary series

$$f(x) = \sum_{n\geq 0} a_n \left(e^{2\pi i\lambda_n x} + e^{-2\pi i\lambda_n x}\right)$$

is differentiable in at least one $x_0 \in \mathbb{S}^1$. Then

$$\lim_{n\to\infty} |a_n\lambda_n| = 0.$$

An immediate consequence of the theorem is the non-differentiability of the Weirstrass fucntion W defined in (4.1). To prove the theorem, we will need the following lemma.

Lemma 4.3. Let $f \in C^0(\mathbb{S}^1)$ satisfy the assumption $f(t) = O(t)$ for sufficiently small $|t|$. Moreover, assume that $n_0 \in \mathbb{Z}$ is an isolated point in the Fourier spectrum of f in the sense that $\widehat{f}(j) = 0$ for each j with $1 \leq |n_0 - j| < 2N$. Then, for some universal constant D, we have

$$\left|\widehat{f}(n_0)\right| \leq D\left[\frac{1}{N}\sup_{0<t<N^{-1/4}}\left|\frac{f(t)}{t}\right| + \frac{1}{N^2}\|f\|_{L^1(\mathbb{S}^1)}\right].$$

Proof. The proof relies on exploiting the vanishing property of f at the origin to introduce a kernel-like function thanks to which we obtain the result. We write

$$\widehat{f}(n_0) = \int_{-1/2}^{1/2} e^{-2\pi i n_0 x} f(x)\,\mathrm{d}x = \int_{-1/2}^{1/2} e^{-2\pi i n_0 x} g_N(x) f(x)\,\mathrm{d}x,$$

and set g_N to be a trigonometric polynomial chosen suitably so that it satisfies the following properties.

(i) $\widehat{g_N}(0) = 1$.

(ii) $\operatorname{supp}\widehat{g_N} \subset (-2N, 2N)$.

Then we may write $g_N(x) - 1 = \sum_{n\neq 0,\ |n|<2N} c_n e^{2\pi i n x}$, and

$$\begin{aligned}\widehat{f}(n_0) &= \int_{-1/2}^{1/2} e^{-2\pi i n_0 x} g_N(x) f(x)\,\mathrm{d}x \\ &= \int_{-1/2}^{1/2} e^{-2\pi i n_0 x} f(x)\,\mathrm{d}x + \int_{-1/2}^{1/2} e^{-2\pi i n_0 x}[g_N(x) - 1] f(x)\,\mathrm{d}x.\end{aligned}$$

Using the expression for $g_N - 1$ and by the assumption that n_0 is isolated,

$$\int_{-1/2}^{1/2} e^{-2\pi i n_0 x} c_n e^{2\pi i n x} f(x)\,\mathrm{d}x = c_n \widehat{f}(n_0 - n) = 0, \quad \text{if } 1 \leq |n| < 2N.$$

We easily see that choosing $g_N := c_N K_N^2(x)$, with c_N such that (i) holds, i.e. normalizing $c_N = \|K_N\|_{L^2(\mathbb{S}^1)}^{-2}$, then (ii) holds true as well. Moreover, we have an explicit expression for K_N^2,

$$K_N^2(x) = \frac{1}{N^2}\left[\frac{\sin(\pi N x)}{\sin(\pi x)}\right]^4 \leq N^{-2}x^{-4},\ x \in (-1/2, 1/2).$$

We also find that

$$c_N^{-1} = \|K_N\|_{L^2(\mathbb{S}^1)}^2 = \sum_{|k|<N}\left(1 - \frac{|k|}{N}\right)^2 \approx dN.$$

Hence we reach the estimate

$$g_N(x) \leq dN^{-3}x^{-4}.$$

Splitting the integral,

$$\widehat{f}(n_0) = \underbrace{\int_{|x|<N^{-1}} e^{-2\pi i n_0 x} g_N(x) f(x)\,\mathrm{d}x}_{I}$$

$$+\underbrace{\int_{N^{-1}<|x|<N^{-1/4}} e^{-2\pi i n_0 x} g_N(x) f(x)\,\mathrm{d}x}_{II} + \underbrace{\int_{|x|>N^{-1/4}} e^{-2\pi i n_0 x} g_N(x) f(x)\,\mathrm{d}x}_{III},$$

we bound these three terms separately,

$$|I| \le N^{-1}\left(\sup_{0<t<|N|^{-1}} \frac{|f(t)|}{|t|}\right) \|g_N\|_{L^1(\mathbb{S}^1)} = N^{-1}\left(\sup_{0<|t|<N^{-1}} \frac{|f(t)|}{|t|}\right),$$

while

$$|II| \le dN^{-3} \int_{N^{-1}<|x|<N^{-1/4}} x^{-4}\,|f(x)|\,\mathrm{d}x \le dN^{-1}\left(\sup_{0<|t|<N^{-1/4}} \frac{|f(t)|}{|t|}\right).$$

Finally,

$$|III| \le dN^{-3} \int_{N^{-1/4}<|x|<1/2} x^{-4}\,|f(x)|\,\mathrm{d}x \le dN^{-2}\,\|f\|_{L^1(\mathbb{S}^1)}\,. \qquad \square$$

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. We reduce to the situation of the lemma with $n_0 = \lambda_n$. First, we need to ensure that $f(t) = O(t)$ for $|t|$ small. A priori, we know that $f(x)$ is differentiable at some $x_0 \in \mathbb{S}^1$. Replacing $f(x)$ by $f(x+x_0)$ does not change the assumptions on the Fourier expansion, since $\widehat{f(\cdot + x_0)}(n)$ gets replaced by $e^{2\pi i x_0 n}\widehat{f}(n)$. We may therefore replace a_n by $a_n e^{2\pi i x_0 n}$. Thus we can assume f is differentiable at $x = 0$. By replacing $f(x)$ by $\tilde{f}(x) = f(x) - f(0)\cos x - f'(0)\sin x$, we still have a lacunary Fourier series for $\tilde{f}$, but this vanishes, along with its first derivative, at $x = 0$. Hence, we may assume that f is differentiable with $f(0) = f'(0) = 0$, and

$$\sup_{0<|x|<N^{-1/4}} \left|\frac{f(x)}{x}\right| \overset{N\to\infty}{\longrightarrow} f'(0) = 0.$$

Finally, setting $n_0 = \lambda_n$, then the next non-zero Fourier coefficients are at λ_{n-1} and λ_{n+1}. By the assumption that this is a lacunary sequence,

$$\lambda_n - \lambda_{n-1} > \lambda_n(1-1/a)\,, \quad \text{and} \quad \lambda_{n+1} - \lambda_n > (a-1)\lambda_n$$

Setting $2N = \min\{(1-1/a)\lambda_n, (a-1)\lambda_n\}$ in the lemma, we find

$$\left|\widehat{f}(\lambda_n)\right| \le C\left[\lambda_n^{-1} \sup_{0<|x|<((1-1/a)\lambda_n)^{-1/4}} \left|\frac{f(x)}{x}\right| + \lambda_n^{-2}\,\|f\|_{L^1(\mathbb{S}^1)}\right] \le D\lambda_n^{-1} o_n(1)$$

where $\lim_{n\to\infty} o_n(1) = 0$, which completes the proof. $\square$

4.1 Weyl's theorem

At this point we consider a question in number theory, where we find connections between Fourier analysis and the distribution of irrational numbers. Indeed, consider $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and the sequence $n\alpha$ where $n = 1, 2, 3, \ldots$ Then $\alpha_n = |n\alpha - \lfloor n\alpha \rfloor| \in (0,1)$ can be interpreted as a point in $\mathbb{S}^1$, and we may ask ourselves what the distribution of α_n in $\mathbb{S}^1$ looks like. Indeed, are these densely distributed? Uniformly distributed? Is the set I of all points α_n an open interval? Does its number of elements depend (perhaps asymptotically) on $|I|$?

Furthermore, the question of what happens to sequences $n^k\alpha$ or polynomial expressions

$$\varphi(n) = \alpha n^k \sum_{j=0}^{k-1} a_j n^j.$$

arises. Hardy and Littlewood conjectured a result for which some time later H. Weyl provided a clean, positive answer. His methods allowed for the treatment of more general questions.

Theorem 4.4. Let $f \in C^0(\mathbb{S}^1)$. Then

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(n\alpha) = \int_{\mathbb{S}^1} f(x)\,\mathrm{d}x \tag{4.2}$$

An interesting, immediate consequence is the following.

Corollary 4.5. Letting $I \subset \mathbb{S}^1$ an open interval, we have

$$\lim_{N\to\infty} \frac{1}{N} \left|\{n \in [1,N] \colon n\alpha - [n\alpha] \in I\}\right| = |I|\,.$$

This corollary of course gives much more than density, as it indicates that the numbers $n\alpha - [n\alpha]$ will spread uniformly in a very precise way over $(0,1)$. We will now prove Theorem 4.4.

Proof of Theorem 4.4. The idea is to use Fourier series, instead of approaching the problem through a pigeonhole point of view (which historically had been the most common method). First, assume that f is a trigonometric polynomial. By the linear dependence of both sides of on f in (4.2), it suffices to consider $f(x) = e^{2\pi ikx}$ for $k \in \mathbb{Z}$.

If $k = 0$, then $f = 1$ and, independently of N, both sides of (4.2) equal 1. If $k \neq 0$, we evaluate both sides explicitly

$$\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi ikn\alpha} = \frac{e^{2\pi ik\alpha}}{N}\sum_{n=0}^{N-1} e^{2\pi ink\alpha} = \frac{e^{2\pi ik\alpha}}{N}\frac{e^{2\pi iNk\alpha}-1}{e^{2\pi ik\alpha}-1} \underset{N\to\infty}{\longrightarrow} 0.$$

At the same time, $\int_{\mathbb{S}^1} f(x) = 0$. Finally, we conclude by density of trigonometric polynomials with respect to the uniform convergence. □

Proof of Corollary 4.5. The only obstacle in deducing the corollary from the theorem is the fact that $\mathbb{1}_I$ is not continuous, but only ***mildly*** so in an appropriate sense. It can be well approximated from above and below by continuous functions. Indeed, given $\varepsilon > 0$ we may find piece-wise linear continuous functions f_1, f_2 such that $f_1(x) \leq \mathbb{1}_I \leq f_2(x)$, with $f_1, f_2 \leq 1$, and

$$\int_{\mathbb{S}^1} |f_1(x) - f_2(x)|\,\mathrm{d}x < \frac{\varepsilon}{3}.$$

Given this setup, consider

$$\left|\frac{1}{M}\sum_{n=1}^{M} \mathbb{1}_I(n\alpha) - |I|\right| \leq \left|\frac{1}{M}\sum_{n=1}^{M} \mathbb{1}_I(n\alpha) - \frac{1}{M}\sum_{n=1}^{M} f_1(n\alpha)\right|$$

$$+ \left| \frac{1}{M} \sum_{n=1}^{M} f_1(n\alpha) - \int_{\mathbb{S}^1} f_1(x)\,\mathrm{d}x \right| + \left| \int_{\mathbb{S}^1} f_1(x)\,\mathrm{d}x - |I| \right|.$$

We may substitute $\mathbb{1}_I$ by f_2 in the first term and choose M large enough to make the second term smaller than $\varepsilon/3$ (thanks to Theorem 4.4). We may estimate the last term by using the inverse triangle inequality and the condition that f_1 approximates $\mathbb{1}_I$ in $L^1(\mathbb{S})$. □

We now come to the much stronger result.

Theorem 4.6 (*Weyl, 1916*). Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $q \geq 1$, $a_j \in \mathbb{R}$ for $j = 0, 1, \ldots, q-1$ and set

$$\varphi(n) := \alpha n^q + \sum_{j=0}^{q-1} a_j n^j.$$

Then the numbers $\{\varphi(n)\}_{n\geq 1} \subset \mathbb{S}^1$ are uniformly distributed.

In order to prove the theorem, we will first reduce to showing that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m \varphi(n)} = 0,$$

for any $m \in \mathbb{Z} \setminus \{0\}$, which in turn is easily reduced to showing the case $m = 1$, as we will see in the sequel.

4.2 Technical preparations

In the proof of Theorem 4.6 we will be led to consider exponential sums of the form

$$\sum_{|\mathbf{r}|\leq n} e^{2\pi i \left(\prod_{j=1}^{q} r_j\right)\alpha},$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_q) \in \mathbb{Z}^q$, we set $|r| := \sum_{i=1}^{q} |r_i|$ and α is an irrational number.

Proposition 4.7. Denote by $n_q := |\{\mathbf{r} \in \mathbb{Z}^q \colon |r| \leq n\}|$. Then,

$$\lim_{n\to\infty} \frac{1}{n_q} \sum_{|\mathbf{r}|\leq n} e^{2\pi i \left(\prod_{j=1}^{q} r_j\right)\alpha} = 0.$$

The proof is carried out by induction, but before we can present it, we need the following lemma.

Lemma 4.8. It holds true that

$$\lim_{n\to\infty} \frac{1}{n_q} \frac{(2n)^q}{q!} = 1.$$

Proof. Note that $n_q \geq C n^q$ for $C = q^{-q}$, and that the subcollection of $\mathbf{r}$ with at least one $r_i = 0$ has cardinality bounded by $D n^{q-1}$. It follows that the number $\tilde{n}_q$ of $\mathbf{r}$ with $|\mathbf{r}| \leq n$ and all r_i

nonzero is asymptotically equal to n_q. Letting

$$A_q^n := \{\mathbf{r}\colon\ |\mathbf{r}| \le n,\ r_i > 0,\ \forall i\},$$

we find that $|A_q^n| = \tilde{n}_q/2^q$, and we may map the set A_q^n bijectively onto the set of increasing q-tuples in $\{1, 2, \dots, n\}$ by considering

$$r_1, r_1 + r_2, \dots, r_1 + r_2 + \cdots + r_q.$$

The latter set having cardinality $\binom{n}{q}$, we find

$$\lim_{n\to\infty} \frac{2^q n^q}{\tilde{n}_q q!} = 1.$$

□

We are now ready to prove Proposition 4.7.

Proof of Proposition 4.7. The case $q = 1$ is essentially contained in the proof of Theorem 4.4. By induction, we may assume the result for $q-1$. Then, following the argument from Corollary 4.5, we infer that

$$\lim_{n\to\infty} \frac{1}{n_{q-1}} \left| \{\mathbf{r}'\} : |\mathbf{r}'| \le n, \left(\prod_{j=1}^{q} r_j\right) \alpha - \left[\left(\prod_{j=1}^{q} r_j\right) \alpha\right] \in J \right| = |J|, \tag{4.3}$$

for any subinterval J of $I = [0, 1]$. Then write

$$\sum_{|\mathbf{r}|\le n} = \sum_{|\mathbf{r}'|\le n} \sum_{|r_q|\le n-|\mathbf{r}'|},$$

where we let $\mathbf{r} = (\mathbf{r}', r_q)$ with $\mathbf{r}' = (r_1, \dots, r_{q-1})$. Now, given any $\varepsilon > 0$, denote the product $R := \prod_{i=1}^{q-1} r_i$, and write

$$\frac{1}{n_q} \sum_{|\mathbf{r}|\le n} e^{2\pi i(\prod_{i=1}^{q} r_i)\alpha} = \frac{1}{n_q} \sum_{|\mathbf{r}'|\le n} \sum_{|r_q|\le n-|\mathbf{r}'|} e^{2\pi i (R r_q)\alpha} = A + B,$$

where for the term A, we restrict the outer sum (over $\mathbf{r}'$) to those $\mathbf{r}'$ with $R\alpha - [R\alpha] \in (0, \varepsilon] \cup [1-\varepsilon, 1)$, while B denotes the sum over the remaining terms. Now, for $\mathbf{r}'$ corresponding to a term in B we get

$$\left| \sum_{|r_q|\le n-|\mathbf{r}'|} e^{2\pi i (R r_q)\alpha} \right| = \left| e^{-2\alpha\pi i R(n-|\mathbf{r}'|)} \frac{e^{2R\alpha\pi i[2(n-|\mathbf{r}'|)+1]} - 1}{e^{2\pi i R\alpha} - 1} \right| \le \frac{1}{|\sin \pi\varepsilon|},$$

ans so we find the bound

$$|B| \le \frac{n_{q-1}}{n_q} \frac{1}{\sin \pi\varepsilon} < \frac{q}{n} \frac{1}{\sin \pi\varepsilon} < \varepsilon$$

for n large enough, where we have taken advantage of the preceding lemma for the second inequality. In order to bound A we use the crude estimate

$$\left| \sum_{|r_q|\le n-|\mathbf{r}'|} e^{2\pi i (R r_q)\alpha} \right| \le 2n + 1,$$

as well as the following consequence of (4.3),

$$\frac{1}{n_{q-1}}\left|\left\{\mathbf{r}':\ \left|\mathbf{r}'\right|\leq n,\ R\alpha-[R\alpha]\in(0,\varepsilon]\cup[1-\varepsilon,1)\right\}\right|<3\varepsilon,$$

provided n is large enough. We conclude that for n large enough,

$$|A|<3\varepsilon\frac{n_{q-1}(2n+1)}{n_q}.$$

After an application of the preceding lemma, together with the bounds for A and B, we conclude that

$$\left|\frac{1}{n_q}\sum_{|r|\leq n}e^{2\pi i(\prod_{i=1}^{q}r_i)\alpha}\right|<\varepsilon(3q+1)$$

for n large enough. Since $\varepsilon>0$ is arbitrary, the proposition follows by induction. □

4.3 Proof of Weyl's theorem

Following the notation in Weyl's original paper [We], we write

$$\sigma_n:=\sum_{k=0}^{n}e^{2\pi i\varphi(k)}.$$

Notice that since $\varphi(k)$ is real, we have

$$\overline{\sigma_n}=\sum_{l=0}^{n}e^{-2\pi i\varphi(l)},$$

and so

$$|\sigma_n|^2=\sum_{k=0}^{n}\sum_{l=0}^{n}e^{2\pi i(\varphi(k)-\varphi(l))}.$$

To set things up more systematically for later, we re-label indices and set $k=h_1$, $l=h_2$, so that $h_1=r_1+h_2$, where each h_j runs over the interval $[0,n]$ while r_1 runs over $[-n,n]$. More precisely, if we now sum over r_1,h_2 instead, then for fixed $r_1\in[-n,n]$, the parameter h_2 will run over a shorter interval $[0,n-|r_1|]$ if $r_1>0$ or $[|r_1|,n]$ if $r_1\leq 0$. Then, writing

$$\varphi(r_1+h_2)-\varphi(h_2)=r_1\varphi(r_1,h_2),$$

we see that $\varphi(r_1,h_2)$ is a polynomial in 2 variables of degree $q-1$, with leading term $q\alpha h_2^{q-1}$ in the variable h_2. Thus we have

$$|\sigma_n|^2=\sum_{r_1\in[-n,n]}\sum_{h_2}^{\sim}e^{2\pi i r_1\varphi(r_1,h_2)},$$

where the second summation is restricted to the sub-intervals $J_1(r_1)$ depending on r_1 described above.

The idea now is to form differences of polynomials, this time of the $\varphi(r_1,h_2)$ with respect to different arguments h_2. For this we have to square things again, and take advantage of the Cauchy-Schwarz inequality:

$$|\sigma_n|^4\leq n_1\sum_{r_1in[-n,n]}\left|\sum_{h_2}^{\sim}e^{2\pi i r_1\varphi(r_1,h_2)}\right|^2,$$

where we recall the notation from the previous section $n_1 = |[-n,n]| = 2n+1$. We can rewrite the preceding as

$$|\sigma_n|^4 \leq n_1 \sum_{r_1 \in [-n,n]} \sum_{h_{2,3}}^{\sim} e^{2\pi i r_1(\varphi(r_1,h_2)-\varphi(r_1,h_3))},$$

where in the inner double sum it is understood that the same restrictions apply to h_2, h_3, i.e. $h_{2,3} \in J_1(r_1)$. Then set $h_2 = r_2 + h_3$. Importantly, since $h_{2,3}$ are restricted to an interval of length $n - |r_1|$ (as implied by $\sum^{\sim}$), we have $|r_2| + |r_1| \leq n$. Furthermore, if $r_2 < 0$ is fixed, then h_3 will be restricted to an even smaller interval, which is obotained from $J_1(r_1)$ by removing $|r_2|$ uints from the left endpoint, while if $r_2 \geq 0$, we remove r_2 units from the right endpoint. Setting

$$\varphi(r_1,h_2) - \varphi(r_1,h_3) = r_2\varphi(r_1,r_2,h_3),$$

we see that $\varphi(r_1,r_2,h_3)$ is a polynomial of degree $q-2$ in three variables, which starts with $\alpha q(q-1)h_3^{q-2}$ in terms of h_3. In all, we now see that we have

$$|\sigma_n|^4 \leq n_1 \sum_{|r_1|+|r_2|\leq n} \sum_{h_3}^{\sim\sim} e^{2\pi i r_1 r_2 \varphi(r_1,r_2,h_3)},$$

where $\sum^{\sim\sim}$ means that for fixed $r_{1,2}$ with $|r_1 + |r_2|| \leq n$, we restrict h_3 to an interval $J_2(r_1,r_2)$ obtained as in the preceding discussion, of lefnth $n - |r_1 - |r_2||$. At this point, it is clear how to continue. We again square the preciding inequality, and invoke Cauchy-Schwarz. We re-index and compute the new intervals.

This process is stopped at stage $q-1$, i.e. after introducing the variable r_{q-1}, and tge $(q-1)$st difference polynomial in q variables,

$$\varphi(r_1,r_2,\ldots,r_{q-1},h_q),$$

which is the a ***linear polynomial*** in these variables, i.e. can be written down in the form

$$\varphi(r_1,r_2,\ldots,r_{q-1},h_q) = q!\alpha h_q + (\beta_0 + \beta_1 r_1 + \cdots + \beta_{q-1}r_{q-1})$$

for certain coefficients β_j which depend on the first two coefficients of φ in a way that is irrelevant to us. Thus, we have the inequality

$$|\sigma_n|^{2^{q-1}} \leq n_1^{2^{q-3}} n_2^{2^{q-4}} \ldots n_{q-2} \sum_{\sum_i |r_i| \leq n} e^{2\pi i \rho} \sum_{h_q}^{q-1} e^{2\pi i R q! \alpha h_q},$$

where we have set $\rho := R(\beta_0 + \beta_1 r_1 + \cdots + \beta_{q-1}r_{q-1})$, and we recall $R = \prod_{i=1}^{q-1} r_i$. According to the lemma, we have

$$n_1^{2^{q-3}} n_2^{2^{q-4}} \ldots n_{q-2} \leq C_q n^{2^{q-3}+2\cdot 2^{q-4}+\cdots+(q-2)\cdot 2^0} = C_q n^{2^{q-1}-q}$$

for suitable C_q. Given any $\varepsilon > 0$, we decompose

$$\sum_{\sum_i |r_i| \leq n} e^{2\pi i \rho} \sum_{h_q}^{q-1} e^{2\pi i R q! \alpha h_q} = A + B,$$

where A comprises those $\mathbf{r}' = (r_1, r_2, \ldots, r_{q-1})$ with the property that

$$Rq!\alpha - [Rq!\alpha] \in (0, \varepsilon] \cup [1 - \varepsilon, 1).$$

According to Proposition 4.7 and its consequence (analogous to Corollary 4.5), we have that the number of $(q-1)$-tuples $\mathbf{r}'$ corresponding to the sum A is of cardinality strictly smaller than $3\varepsilon n_{q-1}$ for large n. And by bounding the inner sum from above by $2n+1$, we infer the bound

$$|A| \leq 3\varepsilon n_{q-1}(2n+1).$$

On the other hand, for B, we get the bound

$$|B| \leq n_{q-1} \frac{1}{\sin \pi\varepsilon}.$$

Taking advantage of the lemma, we find

$$|\sigma_n|^{2^{q-1}} \leq D_q n^{2^{q-1}} \left[3\varepsilon + \frac{1}{n \sin \pi\varepsilon} \right].$$

By the arbitrariness of $\varepsilon > 0$, we find

$$\lim_{n \to \infty} \frac{\sigma_n}{n} = 0$$

as desired.

5 The Fourier transform in $\mathbb{R}^n$

In this chapter we leave the compact setting of $\mathbb{S}^1$ and move onto $\mathbb{R}^n$, where instead of Fourier series, we will have the Fourier transform.

5.1 The Schwartz space

We begin by recalling the ***multi-index*** notation.

Notation. Given $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ a multi-index, we write

$$x^\beta := x_1^{\beta_1} \cdot \dots \cdot x_d^{\beta_d}, \quad \partial^\beta := \partial_1^{\beta_1} \dots \partial_d^{\beta_d}, \quad |\beta| := \sum_{j=1}^{d} \beta_j.$$

The Schwartz space of functions on $\mathbb{R}^n$ comprises all those functions that are -together with their derivatives- rapidly decaying, and through which we will be able to define the Fourier transform appropriately for various $L^p(\mathbb{R}^n)$ spaces.

Definition 5.1. A function $f \in C^\infty(\mathbb{R}^d)$ is said to belong to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ if

$$\sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial^\beta f \right| \leq C, \quad \forall \alpha, \beta \in \mathbb{N}^d.$$

The most basic example of a Schwartz function is the Gaussian

$$f(x) = e^{-|x|^2}, \quad |x|^2 = \sum_{i=1}^{n} x_i^2, \quad x \in \mathbb{R}^n.$$

The exponential decay ensures that $f \in \mathcal{S}(\mathbb{R}^n)$.

Proposition 5.2. Functions $f \in \mathcal{S}(\mathbb{R}^n)$ in the Schwartz space satisfy the following properties.

(i) For every multi-index $\alpha \in \mathbb{N}^d$, $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$.

(ii) If p is a polynomial, then $f \cdot p \in \mathcal{S}(\mathbb{R}^n)$.

(iii) The Schwartz class is dense in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.

The first two properties are easily deduced from the definition, while the third one is a consequence of $C_c^\infty(\mathbb{R}^d)$ living inside $\mathcal{S}(\mathbb{R}^n)$. More importantly, we can define the ***Fourier transform*** on the Schwartz space.

Definition 5.3. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We define its Fourier transform via

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x.$$

Notice that this is well defined whenever $f \in L^1(\mathbb{R}^n)$ or when f is compactly supported. We can immediately deduce a few properties of the Fourier transform that are present in Fourier series.

Proposition 5.4. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, α and β two multi-indices in $\mathbb{N}^d$ and $h \in \mathbb{R}^n$. If $\tau_h f(x) = f(x-h)$ and $m_\alpha(f) = x^\alpha f$, then

(i) $\widehat{\tau_h(f)}(\xi) = e^{2\pi i h\xi}\widehat{f}(\xi)$.

(ii) $\mathcal{F}\big(e^{2\pi i x h}f(x)\big) = \widehat{f}(\xi - h)$.

(iii) $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.

(iv) $\widehat{\partial^\beta f}(\xi) = (2\pi i\xi)^\beta\widehat{f}(\xi)$.

(v) $\mathcal{F}\big((2\pi i x)^\beta f\big)(\xi) = \partial^\beta\widehat{f}(\xi)$.

In the Schwartz class, we have the following.

Lemma 5.5. The Fourier transform is a well-defined operator on $\mathcal{S}(\mathbb{R}^n)$ and moreover $\mathcal{F}(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$.

Proof. We may apply properties (iv) and (v) in Proposition 5.4 to bound

$$(2\pi)^{|\beta|}\left|\xi^\alpha\partial^\beta\widehat{f}(\xi)\right| \leq \int_{\mathbb{R}^n}\left|\partial^\alpha\Big((2\pi i x)^\beta f(x)\Big)\right| e^{-2\pi i x\cdot\xi}\,\mathrm{d}x \leq \int_{\mathbb{R}^n}\frac{C}{(1+|x|)^{n+1}}\,\mathrm{d}x,$$

where we also used the fact that $f \in \mathcal{S}(\mathbb{R}^n)$, and the integral above is bounded. □

Furthermore, within the Schwartz class we have an inversion formula for the Fourier transform.

Theorem 5.6 (*Inversion*). For $f \in \mathcal{S}(\mathbb{R}^n)$,

$$f(x) = \int \widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi.$$

Proof. By plugging in the definition of the Fourier transform, one comes to a dead end. This, however, can be bypassed by approximating the Fourier transform by using a Gaussian function, whose Fourier transform is the Gaussian itself.

Let $g(\xi) = e^{-\pi|\xi|^2}$. Then $g(\varepsilon\xi) = e^{-\pi\varepsilon^2|\xi|^2} \longrightarrow 1$ uniformly over every compact set as $\varepsilon \to 0$. Denote by $\mathcal{F}^*$ the adjoint operator of $\mathcal{F}$, and notice that

$$\mathcal{F}^*\mathcal{F}f(x) = \int\int f(y)e^{-2\pi i\xi\cdot y}\,\mathrm{d}y\,e^{2\pi i\xi\cdot x}\,\mathrm{d}\xi = \lim_{\varepsilon\to 0}\int\left(\int f(y)e^{-2\pi i\xi\cdot y}\,\mathrm{d}y\right)e^{2\pi i\xi\cdot x}g(\varepsilon\xi)\,\mathrm{d}\xi$$

by dominated convergence and since $g \in \mathcal{S}(\mathbb{R}^n)$. Because the function inside the first integral can be bounded by $|\widehat{f}| \in L^1$, this equals

$$\lim_{\varepsilon\to 0}\int f(y)\int g(\varepsilon\xi)e^{2\pi i\xi\cdot(x-y)}\,\mathrm{d}\xi\ \mathrm{d}y \underset{\varepsilon\xi=\eta}{=} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^d}\int f(y)\int g(\eta)e^{2\pi i\eta\cdot\frac{x-y}{\varepsilon}}\,\mathrm{d}\eta\ \mathrm{d}y =$$
$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^d}\int f(y)g\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}y = \lim_{\varepsilon\to 0}\int f(y)K_\varepsilon(x-y)\,\mathrm{d}y = f(x),$$

where we used Fubini's theorem, the fact that $\mathcal{F}^*g = g$ and $K_\varepsilon(z) = (1/\varepsilon^d)g(z/\varepsilon)$ is easily checked to be an approximate identity. □

The following lemma establishes the claim that the Fourier transform of a Gaussian is itself.

Lemma 5.7. Let $g(x) = e^{-\pi|x|^2}$. Then $\widehat{g}(\xi) = g(\xi) = e^{-\pi|\xi|^2}$.

Proof. There are several ways to prove this. The first one is by complex integration.

$$\widehat{g}(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi\cdot x}e^{-\pi|x|^2}\,\mathrm{d}x = \int_{\mathbb{R}} e^{-\pi|x-i\xi|^2}e^{-\pi|\xi|^2}\,\mathrm{d}x = e^{-\pi|\xi|^2}\int_{\mathbb{R}} e^{-\pi|x'|^2}\,\mathrm{d}x' = e^{-\pi|\xi|^2}.$$

Here one must take into account the contribution of g when $|x| \to \infty$ before changing x to x'. However, due to the fast decay of g, that is zero, and integrating over $x - i\xi$ (where $x \in \mathbb{R}$) is the same as integrating over $\mathbb{R}$.

The other way to prove this, which does not involve complex integration, is by noting that g is the solution to

$$\begin{cases} g'(x) = -2\pi x g(x) \\ g(0) = 1. \end{cases}$$

By taking the Fourier transform in the equation, we get

$$(\widehat{g})'(\xi) = -2\pi\xi\widehat{g}(\xi)$$

and we end up with the same differential equation that g satisfies. By computing

$$\widehat{g}(0) = \int_{\mathbb{R}} e^{-\pi|\xi|^2}\,\mathrm{d}\xi = 1,$$

we find that g and $\widehat{g}$ satisfy the same initial condition, and therefore by uniqueness, $g = \widehat{g}$. Furthermore, we can extend this to $\mathbb{R}^n$ by noting that

$$\widehat{g}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}e^{-\pi|x|^2}\,\mathrm{d}x = \int_{\mathbb{R}^n}\left(\prod_{j=1}^n e^{-2\pi i x_j\xi_j}\right)\left(\prod_{k=1}^n e^{-\pi x_k^2}\right)\mathrm{d}x =$$
$$= \prod_{j=1}^n\int_{\mathbb{R}} e^{-2\pi i x_j\xi_j}e^{-\pi x_j^2}\,\mathrm{d}x_j = \prod_{j=1}^n \widehat{g_j}(\xi_j) = \prod_{j=1}^n g_j(x_j) = \prod_{j=1}^d e^{-\pi x_j^2} = g(x). \qquad \square$$

We now move onto the context of L^p-spaces, with a particular emphasis on the case $p = 2$. Moreover, notice that the inversion formula already shows that the adjoint operator $\mathcal{F}^*$ is, in fact, the inverse $\mathcal{F}^{-1}$ on $\mathcal{S}(\mathbb{R}^n)$. In the L^2-product, the consequences of this are a pivotal result called ***Plancherel's formula***.

5.2 The Fourier transform in $L^2(\mathbb{R}^n)$

Much like in the compact case, when $f \in L^2(\mathbb{R}^n)$, the Fourier transform in an isometry.

Theorem 5.8 (*Plancherel*). For $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle f, g \rangle_{L^2} = \left\langle \widehat{f}, \widehat{g} \right\rangle_{L^2}.$$

Proof. It is a direct consequence of the fact that $\mathcal{F}^* = \mathcal{F}^{-1}$,

$$\langle \mathcal{F}f, \mathcal{F}g \rangle = \langle f, \mathcal{F}^*\mathcal{F}g \rangle = \langle f, g \rangle. \qquad \square$$

This implies that the Fourier transform extends uniquely from the Schwartz space into the Hilbert space of square integrable functions.

Corollary 5.9. There exists a unique linear isometry $\mathcal{F}\colon L^2 \longrightarrow L^2$ which extends $\mathcal{F}$ from $\mathcal{S}(\mathbb{R}^n)$ to L^2. Moreover, $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = \mathrm{id}$.

Proof. The proof follows a density argument. Indeed, $\mathcal{F}$ is bounded and linear as an operator from $\mathcal{S}(\mathbb{R}^n)$ into itself. Given $f \in L^2$, we may approximate it by a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ in the L^2-norm. Then,

$$\left\| \widehat{f_k} - \widehat{f_l} \right\|_{L^2} = \| f_k - f_l \|_{L^2} \underset{j,k\to\infty}{\longrightarrow} 0,$$

thanks to Plancherel's formula. Hence $\widehat{f_k}$ converges to a function which we define to be $\widehat{f}$. $\square$

Notice that, although $\mathcal{F}$ is not explicitly given when $f \in L^2$, we do in fact have an explicit expression whenever $f \in L^1 \cap L^2$, since $\widehat{f_k}$ actually converges in the pointwise sense to $\widehat{f}$.

5.3 The Fourier transform in $L^p(\mathbb{R}^n)$

Recall that $\widehat{f}$ is well defined in $L^\infty(\mathbb{R}^n)$ whenever $f \in L^1(\mathbb{R}^n)$. The topic of L^p convergence of Fourier transforms is more subtle, but this relation suggests that we should look for Hölder conjugate exponents. In fact, the case $p = q = 2$ holds in this sense as well, and we may therefore try to approach this question of convergence in terms of interpolation operators.

Applying Theorem 1.11 to the Fourier transform, taking into account that $\mathcal{F}$ satisfies the hypothesis in the theorem, and taking $q_0 = \infty, p_0 = 1, q_1 = p_1 = 2$, we find that $\mathcal{F}$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ whenever p and q are Hölder conjugate. I.e. we have the following result.

Corollary 5.10 (*Hausdorff-Young inequality*). The Fourier transform $\mathcal{F}$ is a well-defined operator between $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ for $1 \le p \le 2$, and

$$\|\mathcal{F}f\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

6 The Hilbert Transform

The Hilbert transform will be our first example on ***singular integral operators***. By definition, for f "nice enough", which in our case means that $f \in C_0^\infty(\mathbb{R})$, we set, for $\varepsilon > 0$,

$$H_\varepsilon f(x) := \int_{-\infty}^{-\varepsilon} \frac{f(x-y)}{y}\,\mathrm{d}y + \int_{\varepsilon}^{\infty} \frac{f(x-y)}{y}\,\mathrm{d}y = \int_{\varepsilon}^{\infty} \frac{f(x-y)-f(x+y)}{y}\,\mathrm{d}y.$$

The above expression makes sense as $\varepsilon \to 0$, since

$$\lim_{y\to 0} \frac{f(x-y)-f(x+y)}{y} = -2f'(x),$$

and so the function $y \mapsto \frac{f(x-y)-f(x+y)}{y}$ extends continuously to $y = 0$, and is of compact support. Hence the limit can be interpreted as the standard Riemann integral of a continuous function. We denote

$$Hf(x) := \lim_{\varepsilon\to 0} H_\varepsilon f(x)$$

and define it to be the ***Hilbert transform*** of f. Notice moreover that Hf is a C^∞-function if $f \in C_0^\infty(\mathbb{R})$. The interest in this operator lies in the remarkable characterization of Hf in terms of the Fourier transform on $\mathbb{R}$.

Remark. Notice that the Hilbert transform is defined in the sense of the principal value,

$$Hf(x) = \int_{\text{P.V.}} \frac{f(x-y)}{y}\,\mathrm{d}y.$$

We use this notation whenever the function inside the integral has a singularity that, a priori, could pose as an impediment to integrability.

Lemma 6.1. Assume that $f \in C_0^\infty(\mathbb{R})$. Then,

$$\mathcal{F}(Hf)(\xi) = -\pi i\,\mathrm{sign}(\xi)\widehat{f}(\xi).$$

Proof. The proof follows by direct computation, the only technical point being that

$$Hf(x) = \int_{\text{P.V.}} \frac{f(x-y)}{y}\,\mathrm{d}y = \int_{\text{P.V.}} \frac{f(y)}{x-y}\,\mathrm{d}y = O\left(\frac{1}{|x|}\right) \in C^\infty(\mathbb{R}) \cap L^p(\mathbb{R}),\ p > 1.$$

We write

$$\begin{aligned}\mathcal{F}(Hf)(\xi) &= \lim_{N\to\infty}\int_{|x|\le N}\int_{\text{P.V.}} e^{-2\pi i x\xi}\frac{f(y)}{x-y}\,\mathrm{d}y\,\mathrm{d}x\\ &= \lim_{N\to\infty}\int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}\int_{\text{P.V.}} \mathbb{1}_{\{|x|\le N\}}\frac{e^{-2\pi i(x-y)\xi}}{x-y}\,\mathrm{d}x\,\mathrm{d}y\\ &= -\pi i\,\mathrm{sign}(\xi)\int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}\,\mathrm{d}y = -i\pi\,\mathrm{sign}(\xi)\widehat{f}(\xi),\end{aligned}$$

where we split $e^{-2\pi i(x-y)\xi} = \cos(2\pi(x-y)\xi) - i\sin(2\pi(x-y)\xi)$ and noticed that

$$\int_{\text{P.V.}} \mathbb{1}_{\{|x|\le N\}}\frac{\cos(2\pi(x-y)\xi)}{x-y}\,\mathrm{d}x = O\left(\frac{1}{N}\right)$$

in order to reduce the inner integral before to obtain

$$\lim_{N\to\infty}\int_{\text{P.V.}} \mathbb{1}_{\{|x|\le N\}}\frac{e^{-2\pi i(x-y)\xi}}{x-y}\,\mathrm{d}x = (-2i)\int_0^\infty \frac{\sin(2\pi x\xi)}{x}\,\mathrm{d}x = -i\pi\,\mathrm{sign}(\xi). \qquad \square$$

We immediately have an explicit value for the L^2-norm of the Hilbert transform as a consequence of Plancherel's theorem.

Corollary 6.2. Whenever $Hf, f \in L^2$, we have

$$\|Hf\|_{L^2} = \pi\,\|f\|_{L^2}\,.$$

The proof follows by computing using Lemma 6.1 and Plancherel's theorem. Furthermore, in a similar fashion to the Fourier transform, we may now extend the Hilbert transform to $L^2(\mathbb{R})$.

Corollary 6.3. The Hilbert transform extends to an isometry, up to a factor π, as an operator $H\colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$.

Proof. For general $f \in L^2(\mathbb{R})$, approximate f by a sequence $\{f_n\}$ of $C_0^\infty(\mathbb{R})$-functions in the L^2-norm. Then, we know that Hf_n is defined, and

$$\left\|\widehat{Hf_m} - \widehat{Hf_n}\right\|_{L^2} = \pi\,\|f_m - f_n\|_{L^2} \to 0.$$

Hence $\{Hf_n\}$ is a Cauchy sequence, and so it converges to $Hf \in L^2$. $\square$

6.1 L^p bounds for the Hilbert transform

The Hilbert transform is useful in solving the issue of convergence of $S_N f \to f$ in the context of Fourier series, by way of its relevance to the convergence of truncated Fourier integrals. Indeed, consider the inversion formula for the Fourier transform. A natural question we may ask is whether or not this formula holds in contexts that are more general than the spaces L^2 and $\mathcal{S}(\mathbb{R}^n)$. To this end, we define

$$\mathcal{F}^{-1}(\widehat{f}) = \int_{\mathbb{R}^n}\widehat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi,$$

and its truncation

$$\mathcal{F}^{-1}_{\leq N}(\widehat{f}) = \int_{|\xi|\leq N} \widehat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi.$$

For fixed N, this is well-defined as long as $\widehat{f} \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. By Young's inequality, this holds as well for $f \in L^{p'}(\mathbb{R}^n)$, where $p' \in [1, 2]$.

Now, if $p' \in [1, 2]$, then $\widehat{f} \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, we would like to verify whether or not it holds that

$$\lim_{N\to\infty} \left\|\mathcal{F}^{-1}_{\leq N}(\widehat{f}) - f\right\|_{L^{p'}(\mathbb{R}^n)} = 0.$$

There is already an easy case: $p = 2$, given by Plancherel's theorem,

$$\lim_{N\to\infty} \left\|\mathbb{1}_{\{|\xi|\geq N\}}\widehat{f}\right\|_{L^2} = 0.$$

However, when $p' \in [1, 2)$, the answer actually depends on the dimension. For instance, in dimension $n = 1$, this is positively answered for $p' \in (1, 2]$. In dimension $n \geq 2$, the answer is negative, since the Fourier multiplier is not so well-behaved.

Remarkably, for dimension $n = 1$, the L^p boundedness properties of the Fourier transform are closely related to the Hilbert transform. In particular, the restriction to the disk $\{|\xi| \leq N\}$ is simply the restriction to an interval, and the Hilbert transform is simply a restriction to a sign profile, which can decompose the characteristic function $\mathbb{1}_{\{\xi\leq N\}}$ through shifts and linear combinations. We therefore find an expression for $\mathcal{F}$ in terms of H, and the boundedness of H in L^p translates to that of $\mathcal{F}$.

Theorem 6.4. Let $1 < p < \infty$. Then there is $C_p \in \mathbb{R}_+$ such that

$$\|Hf\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}, \quad \forall f \in C_0^\infty(\mathbb{R}).$$

In particular, H extends boundedly to $L^p(\mathbb{R})$.

Observe that this is consistent with the $O(|x|^{-1})$-decay of Hf, and the endpoint $p = 1$ is easily seen to fail generically. We can also make the constant C_p explicit to find that $C_p = C\frac{p}{p-1}$ is suitable, although not sharp.

The usual approach to proving the theorem is through Calderón-Zygmund theory, but in our case we will take a more classical approach, following a proof by Grafakos. We will need the following lemma.

Lemma 6.5. Assuming $C_p \in \mathbb{R}_+$ as in Theorem 6.4, we have

$$C_{2p} \leq \sqrt{2\pi^2 + 4C_p^2}.$$

Proof. This relies on the following identity

$$[Hf]^2 = \pi^2 f^2 + 2H(fHf). \tag{6.1}$$

To see this, we compute the Fourier transform on both sides, denoting $m(\xi) = -i\pi\,\mathrm{sign}(\xi)$. This yields

$$\mathcal{F}(\pi^2 f^2 + 2H(fHf)) = \pi^2(\widehat{f} * \widehat{f})(\xi) + 2m(\xi)(\widehat{f} * \widehat{Hf})(\xi)$$

$$= \pi^2 \int_{\mathbb{R}} \widehat{f}(\xi-\eta)\widehat{f}(\xi)\,\mathrm{d}\eta + m(\xi)\int_{\mathbb{R}} \widehat{f}(\xi-\eta)\widehat{f}(\eta)m(\xi-\eta)\,\mathrm{d}\eta$$
$$+ m(\xi)\int_{\mathbb{R}} \widehat{f}(\xi-\eta)\widehat{f}(\eta)m(\eta)\,\mathrm{d}\eta$$
$$= \int_{\mathbb{R}} \widehat{f}(\xi-\eta)\widehat{f}(\eta)\left[\pi^2 + m(\xi)m(\eta) + m(\xi)m(\xi-\eta)\right]\mathrm{d}\eta.$$

We may use the identity

$$\pi^2 + m(\xi)m(\eta) + m(\xi)m(\xi-\eta) = m(\eta)m(\xi-\eta), \quad \text{a.e.}$$

to simplify the expression in brackets, thanks to which one concludes that this is

$$\int m(\xi-\eta)\widehat{f}(\xi-\eta)m(\eta)\widehat{f}(\eta)\,\mathrm{d}\eta = \widehat{Hf} * \widehat{Hf}.$$

Having shown this identity, we use an induction scheme to show the estimate in the statement. We know that $C_2 = \pi$, since

$$\mathcal{F}f(\xi) = m(\xi)\widehat{f}(\xi) = -i\pi\,\mathrm{sign}(\xi)\widehat{f}(\xi).$$

Hence Plancherel implies the bound in the base case. Now we use the indentity to show the inductive step. Assuming the validity of the result for $p = 2^{k-1}$, we compute

$$\left\|(Hf)^2\right\|_{L^p} = \left(\int_{\mathbb{R}} |Hf|^{2p}\right)^{1/p} = \|Hf\|_{L^{2p}}^2 \le \pi^2\|f\|_{L^{2p}}^2 + 2\|H(f\cdot Hf)\|_{L^p}$$
$$\le \pi^2\|f\|_{L^{2p}}^2 + 2C_p\cdot\|fHf\|_{L^p} \le \pi^2\|f\|_{L^{2p}}^2 + 2C_p\|f\|_{L^{2p}}\|Hf\|_{L^{2p}}.$$

We now use the inequality $2ab \le 2a^2 + \frac{1}{2}b^2$ to find that

$$2C_p\|f\|_{L^{2p}}\|Hf\|_{L^{2p}} \le 2C_p^2\|f\|_{L^{2p}}^2 + \frac{1}{2}\|Hf\|_{L^{2p}}^2.$$

Hence we can bound

$$\frac{1}{2}\|Hf\|_{L^{2p}}^2 \le \pi^2\|f\|_{L^{2p}}^2 + 2C_p^2\|f\|_{L^{2p}}^2,$$

and

$$C_{2p} \le \sqrt{2\pi^2 + 4C_p^2}.$$

□

We are now ready to show Theorem 6.4.

Proof of Theorem 6.4. Using the lemma, we know that for all $p = 2^{k-1}$, the constants $C_{2p} < \infty$ and the result holds true. Next, if $2^k \le p < 2^{k+1}$, we employ the Riesz-Thorin interpolation theorem to obtain boundedness in this range. We get

$$C_p \le C_2^{\theta}C_{2^{k+1}}^{1-\theta}$$

for suitable $\theta \in [0,1]$, and thus H is L^p-bounded for each $p \in [2,\infty)$. Finally, as for the Hölder dual range of $p \in (1,2]$, write

$$\|Hf\|_{L^p} = \sup_{\|g\|_{L^{p'}}\le 1}\left|\int H(f)\overline{g}\right| = \sup_{\|g\|_{L^{p'}}\le 1}\frac{1}{\pi^2}\left|\int \overline{H^2(g)}Hf\right|,$$

since $H^2 g = -\pi^2 g$ by Lemma 6.1. Now, because $\|Hf\|_{L^2} = \pi \|f\|_{L^2}$ and H is linear, we can deduce that

$$\int \overline{Hg} Hf = \pi^2 \int \overline{g} f$$

and

$$\|Hf\|_{L^p} = \sup_{\|g\|_{L^{p'}} \leq 1} \left| \int \overline{Hg} f \right| \leq \sup C'_p \|g\|_{L^{p'}} \|f\|_{L^p} \leq C'_p \|f\|_{L^p},$$

since $p' \geq 2$. □

Let us now move back to the setting of truncated Fourier series and the convergence of $S_N f$ in $L^p(\mathbb{S}^1)$.

Theorem 6.6. Let $1 < p < \infty$ and $f \in L^p(\mathbb{S}^1)$. Then $S_N f$ converges to f in $L^p(\mathbb{S}^1)$,

$$\lim_{N \to \infty} \|S_N f - f\|_{L^p(\mathbb{S}^1)} = 0.$$

Proof. Given Proposition 3.3, it is enough to show that the operator norm $\sup \|S_N\|_{L^p \to L^p} < \infty$. Recall that

$$S_N f(x) = \int_{-1/2}^{1/2} \frac{\sin([2N+1]\pi(x-y))}{\sin\,[\pi(x-y)]} f(y)\,\mathrm{d}y.$$

By density of $C_0^\infty((-\frac{1}{2}, \frac{1}{2})) \subset L^p((-\frac{1}{2}, \frac{1}{2}))$ for all $1 < p < \infty$, it suffices to consider $f \in C_0^\infty((-\frac{1}{2}, \frac{1}{2}))$. We can use the above formula to redefine $S_N f(x)$ as a principal value,

$$S_N f(x) = \lim_{\varepsilon \to 0} \int_{\substack{-1/2 \\ y \in [x-\varepsilon, x+\varepsilon]}}^{1/2} \frac{\sin([2N+1]\pi(x-y))}{\sin[\pi(x-y)]} f(y)\,\mathrm{d}y.$$

Next, replace $\sin[\pi(x-y)]$ by $\pi(x-y)$ -up to a constant- and notice that the function

$$L_N(x-y) = \frac{\sin((2N+1)\pi(x-y))}{\sin[\pi(x-y)]} - \frac{\sin((2N+1)\pi(x-y))}{\pi(x-y)}$$

is bounded. Hence,

$$\left\| \int_{-1/2}^{1/2} L_N(x-y) f(y)\,\mathrm{d}y \right\|_{L^p(\mathbb{S}^1)} \leq \|f\|_{L^p(\mathbb{S}^1)} \|L_N\|_{L^1(\mathbb{S}^1)} \leq C \|f\|_{L^p(\mathbb{S}^1)},$$

where C is a universal constant that is independent of N. Hence we reduce to proving L^p-boundedness of

$$\int_{\substack{-1/2 \\ \text{P.V.}}}^{1/2} \frac{\sin([2N+1]\pi(x-y))}{\pi(x-y)} f(y)\,\mathrm{d}y = \frac{1}{2\pi i} \int_{\substack{-\infty \\ \text{P.V.}}}^{\infty} \frac{\sum_{\pm} e^{\pm i[2N+1]\pi(x-y)}}{x-y} f(y)\,\mathrm{d}y$$

$$= \sum_{\pm} \frac{1}{2\pi i} \int_{\substack{-\infty \\ \text{P.V.}}}^{\infty} \frac{e^{\pm i[2N+1]\pi(x-y)}}{x-y} f(y)\,\mathrm{d}y.$$

Then for either sign $\pm$, write

$$G_{\pm} f := \int_{\substack{-\infty \\ \text{P.V.}}}^{\infty} \frac{f(y)}{x-y} e^{\pm i(2N+1)(x-y)}\,\mathrm{d}y.$$

Using that multiplication by $e^{\pm i(2N+1)x}$ is an isometry in L^p for $x \in \mathbb{R}$, we find

$$\|G_\pm f\|_{L^p((1/2,1/2))} \leq \|G_\pm f\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}$$

with C_p as in the proof of Theorem 6.4. □

We also have an analogous result in the non-compact setting involving Fourier multipliers.

Theorem 6.7. Let $1 < p \leq 2$. Then we have

$$\lim_{N\to\infty} \left\| \int_{|\xi|\leq N} \widehat{f}(\xi) e^{2\pi i(\cdot)\xi}\, \mathrm{d}\xi - f \right\|_{L^p(\mathbb{R})} = 0$$

for each $f \in L^p(\mathbb{R})$.

Proof. This is again done by reduction to H:

$$\int_{|\xi|\leq N} \widehat{f}(\xi) e^{2\pi i x\xi}\, \mathrm{d}\xi = \int \widehat{f}(\xi) e^{2\pi i x\xi} \mathbb{1}_{\{|\xi|\leq N\}}\, \mathrm{d}\xi,$$

and

$$\mathbb{1}_{\{|\xi|\leq N\}} = \frac{1}{4}(\mathrm{sign}(N-\xi)+1)(\mathrm{sign}(N+\xi)+1)\,, \quad \text{for } \xi \notin \{N, -N\}\,.$$

Futher, the multipliers $\mathrm{sign}(N-\xi)$ act boundedly in L^p,

$$\left\| \int e^{2\pi i x\xi}\, \mathrm{sign}(N-\xi)\widehat{f}(\xi)\, \mathrm{d}\xi \right\|_{L^p} \leq \frac{C_p}{\pi} \|f\|_{L^p(\mathbb{R})}\,,$$

where C_p is the constant for H. Then

$$\int \widehat{f}(\xi) e^{2\pi i x\xi} \mathbb{1}_{\{|\xi|\leq N\}}\, \mathrm{d}\xi \leq \frac{C_p^2}{\pi^2} \|f\|_{L^p(\mathbb{R})}\,,$$

and thus

$$\int_{|\xi|\leq N} \widehat{f}(\xi) e^{2\pi i x\xi}\, \mathrm{d}\xi \leq (AC_p + BC_p^2) \|f\|_{L^p}\,.$$

To finish off the proof, given $f \in L^p(\mathbb{R})$, $1 < p \leq 2$, approximate f by a function $g \in \mathcal{S}(\mathbb{R})$ with compact Fourier support. Then,

$$\begin{aligned}&\int_{|\xi|\leq N} \widehat{f}(\xi) e^{2\pi i x\xi}\, \mathrm{d}\xi - f(x) \\ &\qquad = \int_{|\xi|\leq N} \widehat{g}(\xi) e^{2\pi i x\xi}\, \mathrm{d}\xi - g(x) + \int_{|\xi|\leq N} (\widehat{f-g}) e^{2\pi i x\xi}\, \mathrm{d}\xi + (g-f)(x)\end{aligned}$$

and notice that since $g \in \mathcal{S}(\mathbb{R})$, the first term converges to zero in L^p as $N \to \infty$, while the two last terms converge to zero as $f \to g$ in L^p. □

In dimensions $n \geq 2$, the Fourier multipliers of the ball $\mathbb{1}_{|\xi|\leq N}$ no longer behave boundedly in L^p for any $p \leq 2$. This is a celebrated result by Charless Fefferman in 1971, and one of the resasons for which he received the Fields medal in 1978. One could, of course, consider a smooth version of this cutoff, instead of the sharp cutoff. In this case, it is a well understood problem in dimension $n = 2$, but still not completely determined in dimensions $n \geq 3$. The general conjecture is known as the ***Bochner-Riesz*** conjecture.

7 Calderón-Zygmund operators

We now begin with the theory of Calderón-Zygmund operators, which in particular, is useful when treating a variety of operators that appear often in the context of PDEs and fluid dynamics.

Definition 7.1 (*Calderón-Zygmund operator*). We say that a function $K\colon \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{C}$ is a ***Calderón-Zygmund kernel*** if it satisfies the following three conditions.

(i) ***Uniform boundedness***. It holds that $|K(x)| \leq B\,|x|^{-n}$.

(ii) ***Hörmander condition***. For all $y \in \mathbb{R}^n \setminus \{0\}$,

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\,\mathrm{d}x \leq B.$$

(iii) ***Vanishing condition***. For all $r, s > 0$, we have

$$\int_{r<|x|<s} K(x)\,\mathrm{d}x = 0.$$

Notice that the first condition expresses the fact that K is barely not in $L^1(\mathbb{R}^n)$. In dimension $n = 1$, the function $K(x) = 1/x$ is a Calderón-Zygmund Kernel.

Definition 7.2. We say that K is a ***strong*** Calderón-Zygmund kernel if, in addition to being a kernel, it satisfies the bound

$$|\nabla K(x)| \leq \frac{B}{|x|^{n+1}},$$

and $K \in C^1(\mathbb{R}^n \setminus \{0\})$.

The fact that these are ***strong*** lies in that they automatically satisfy the Hörmander condition.

Lemma 7.3. If K is a ***strong*** Calderón-Zygmund kernel, it satisfies the Hörmander condition.

Proof. Indeed, using the fundamental theorem of calculus, write

$$K(x) - K(x-y) = -\int_0^1 \nabla_x K(x-ty)\cdot y\,\mathrm{d}t.$$

For $|y| < |x|/2$, we have

$$\int_{|x|>2|y|} |K(x-y) - K(x)| \, \mathrm{d}x \leq \int_{|x|>2|y|} B\,|y| \left(\frac{|x|}{2}\right)^{-(n+1)} \mathrm{d}x \leq \tilde{B}\,|y|\,|y|^{-1} = \tilde{B}. \qquad \square$$

Definition 7.4. Let K be a Calderón-Zygmund kernel. Then the associated Calderón-Zygmund operator Tf with $f \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon < |x-y|} K(x-y) f(y) \, \mathrm{d}y.$$

We may think of this as the generalization of the principal value in the 1-dimensional context.

Lemma 7.5. The function $Kf(x)$ is defined for each $x \in \mathbb{R}^n$, and it is a C^∞-function for f as above.

Proof. We start by writing

$$T_\varepsilon f(x) := \int_{\mathbb{R}^n} \mathbb{1}_{\{|x-y|>\varepsilon\}} K(x-y) f(y) \, \mathrm{d}y = \int_{\mathbb{R}^n} \mathbb{1}_{\{|y|>\varepsilon\}} K(y) f(x-y) \, \mathrm{d}y.$$

Notice that away from $y = 0$, and since $f \in \mathcal{S}(\mathbb{R}^n)$, we will not have integrability problems. At the origin, however, we may take advantage of the cancellation property of K to write

$$T_\varepsilon f = \int_{\mathbb{R}^n} \mathbb{1}_{\{\varepsilon<|y|<1\}} K(y) \left[f(x-y) - f(x)\right] \mathrm{d}y + \int_{\mathbb{R}^n} \mathbb{1}_{\{|y|>1\}} K(y) f(x-y) \, \mathrm{d}y.$$

And the first integral can be tackled thanks to the fact that

$$|K(y)|\,|f(x-y) - f(x)| \leq B \left\|\nabla f\right\|_{L^\infty} |y|^{-n+1},$$

which is integrable near the origin. Hence $\lim_{\varepsilon \to 0} T_\varepsilon f$ exists. Smoothness can be seen by differentiating under the integral sign. $\square$

Our current goal is to extend T to linear function spaces, namely $L^p(\mathbb{R}^n)$. To this end, we need to deduce uniform L^p bounds for functions $f \in \mathcal{S}(\mathbb{R}^n)$, and we may use a density argument to define T on L^p. Notice that by contrast to H, we cannot use "veiled" complex analysis arguments. In order to achieve our goal, we introduce a particularly interesting tool: the ***Calderón-Zygmund decomposition***.

The strategy for obtaining general L^p-bounds will be as follows.

(i) For L^2-bounds, we may use Plancherel's result to obtain a useful characterization of T on the Fourier side. This is similar to the Hilbert transform.

(ii) We obtain a ***weak*** L^1-bound for T by means of the Calderón-Zygmund decomposition of f.

(iii) Finally, we make use of the Marcinkiewicz interpolation result in order to show L^p-boundedness of T.

Theorem 7.6. Let K be a Calderón-Zygmund kernel and T its associated operator. Then, for each $\varepsilon > 0$ there is $C = C(n) > 0$ such that

$$\|T_\varepsilon f\|_{L^2(\mathbb{R}^n)} \leq CB \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

As a consequence, we immediately obtain the canonical way of extending T to L^2.

Corollary 7.7. The operator T is defined canonically for $f \in L^2(\mathbb{R}^n)$.

Proof. By passing to $\lim_{\varepsilon\to 0}$ in the inequality given by Theorem 7.6, we infer that

$$\|Tf\|_{L^2(\mathbb{R}^n)} \leq CB \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Then, for a general $g \in L^2(\mathbb{R}^n)$, we approximate g by $f_k \in \mathcal{S}(\mathbb{R}^n)$ in the L^2-norm. Then Tf_k form a Cauchy sequence in $L^2(\mathbb{R}^n)$, and we let $Tg := \lim_{k\to\infty} Tf_k$. □

We now show boundedness in L^2 (Theorem 7.6).

Proof of Theorem 7.6. Define

$$T_{r,s} f := \int_{\mathbb{R}^n} \mathbb{1}_{\{r<|y|<s\}} K(y) f(x-y)\, \mathrm{d}y,$$

and notice that $T_\varepsilon f = \lim_{s\to\infty} T_{\varepsilon,s} f$, so we may simply work with $T_{r,s}$ instead of T_ε. We use a ***Fourier interpretation*** of $T_{r,s}$,

$$T_{r,s} f(x) = \int K(y) \mathbb{1}_{\{r<|y|<s\}} f(x-y)\, \mathrm{d}y = \left(\left(K\mathbb{1}_{\{r<|y|<s\}}\right) * f\right)(x).$$

Hence,

$$\mathcal{F}(T_{r,s} f)(\xi) = \mathcal{F}\left(K\mathbb{1}_{\{r<|y|<s\}}\right)(\xi) \widehat{f}(\xi).$$

By Plancherel,

$$\|T_{r,s} f\|_{L^2} = \|\mathcal{F}(T_{r,s} f)\|_{L^2} \leq \left\|\mathcal{F}(K\mathbb{1}_{\{r<|y|<s\}})\right\|_{L^\infty} \|\widehat{f}\|_{L^2} = \left\|\mathcal{F}(K\mathbb{1}_{\{r<|y|<s\}})\right\|_{L^\infty} \|f\|_{L^2}.$$

To bound the L^∞-norm, we split the function inside into two parts: one with oscillatory behavior, and another one with appropriate decay properties. Notice that

$$\begin{aligned}
\mathcal{F}\left(K\mathbb{1}_{\{r<|y|<s\}}\right)(\xi) &= \int K(y) \mathbb{1}_{\{r<|y|<s\}} e^{-2\pi i y\cdot\xi}\, \mathrm{d}x \\
&= \int_{r<|y|<\min\left\{|\xi|^{-1},s\right\}} K(y) e^{-2\pi i y\xi}\, \mathrm{d}y + \int_{\max\left\{r,|\xi|^{-1}\right\}<|y|<s} K(y) e^{-2\pi i y\xi}\, \mathrm{d}y = I_1 + I_2.
\end{aligned}$$

For I_1, we rewrite the integral using the properties of K by introducing a term $-K$ by exploiting the vanishing condition, so as to obtain

$$\int K(y) \mathbb{1}_{\left\{r<|y|<\min\left\{|\xi|^{-1},s\right\}\right\}} e^{-2\pi i y\cdot\xi}\, \mathrm{d}y = \int K(y) \mathbb{1}_{\left\{r<|y|<\min\left\{|\xi|^{-1},s\right\}\right\}} [e^{-2\pi i y\cdot\xi} - 1]\, \mathrm{d}y,$$

where the last factor can be bounded in terms of $|y|\,|\xi|$. Then,

$$|I_1| \leq C_1 \int |K(y)|\, \mathbb{1}_{\left\{r<|y|<\min\left\{|\xi|^{-1},s\right\}\right\}} |y|\,|\xi|\, \mathrm{d}x$$

$$\leq C_1 \int_{r<|y|<\min\left\{s,|\xi|^{-1}\right\}} B\,|y|^{-n+1}\,|\xi|\,\mathrm{d}y \leq C_1 C_2(n) B\,|\xi|^{-1}\,|\xi| = C(n)B.$$

For I_2, we exploit the Hörmander condition (ii) on the kernel K. Write

$$I_2 = \int_{\max\left\{r,|\xi|^{-1}\right\}<|y|<s} K(y)e^{-2\pi i y\cdot\xi}\,\mathrm{d}y = -\int_{\max\left\{r,|\xi|^{-1}\right\}<|y|<s} K(y)\,e^{-2\pi i\left(y+\frac{\xi}{2|\xi|^2}\right)\cdot\xi}\,\mathrm{d}y,$$

whence

$$\begin{aligned} 2I_2 &= \int_{\max\left\{r,|\xi|^{-1}\right\}<|y|<s} \left[K(y) - K\left(y - \frac{\xi}{2\,|\xi|^2}\right)\right] e^{-2\pi i y\cdot\xi}\,\mathrm{d}y \\ &\quad - \int_{X_1\setminus X_2} K\left(y - \frac{\xi}{2\,|\xi|^2}\right) e^{-2\pi i y\cdot\xi}\,\mathrm{d}y + \int_{X_2\setminus X_1} K\left(y - \frac{\xi}{2\,|\xi|^2}\right) e^{-2\pi i y\cdot\xi}\,\mathrm{d}y, \end{aligned}$$

where

$$X_1 \coloneqq \left\{\max\left\{r,|\xi|^{-1}\right\} < \left|y - \frac{\xi}{2\,|\xi|^2}\right| < s\right\}, \quad X_2 = \left\{\max\left\{r,|\xi|^{-1}\right\} < |y| < s\right\}.$$

Then we need to estimate each of these terms. Notice that the first one is bounded by B as per condition (ii) defining K. The remaining two integrals can be dealt with in a similar manner. For the first of these two, notice that the region $X_1 \setminus X_2$ implies that either

$$\left|y - \frac{\xi}{2\,|\xi|^2}\right| \geq -\frac{1}{2}\,|\xi|^{-1} + s \geq \frac{1}{2}s,$$

or

$$\left|y - \frac{\xi}{2\,|\xi|^2}\right| < \max\left\{r,|\xi|^{-1}\right\} + \frac{1}{2}\,|\xi|^{-1} \leq \frac{3}{2}\max\left\{r,|\xi|^{-1}\right\}.$$

In either case, there is a number $\gamma > 0$ such that $\left|y - \frac{\xi}{2|\xi|^2}\right| \in [\gamma, 2\gamma]$. But then, using the boundedness condition (i) on K, we obtain

$$\int_{X_1\setminus X_2} K\left(y - \frac{\xi}{2\,|\xi|^2}\right) e^{-2\pi i y\cdot\xi}\,\mathrm{d}y \leq B \int_{\left|y-\frac{\xi}{2|\xi|^2}\right|\in[\gamma,2\gamma]} \left|y - \frac{\xi}{2\,|\xi|^2}\right|^{-n} \mathrm{d}y \leq B_1.$$

Finally, the remaining integral can be dealt with similarly. □

The key to obtaining L^p bounds for $1 < p < \infty$ now consists in establishing a ***weak*** L^p-bound at the endpoint. To motivate this, look back at the Hilbert transform,

$$Hf(x) = \int_{\text{P.V.}} \frac{f(y)}{x-y}\,\mathrm{d}y.$$

Generically, even when $f \in C_c^\infty(\mathbb{R})$, the best bound for large $|x|$ is

$$|Hf(x)| \leq \frac{C}{|x|} \notin L^1(\mathbb{R}).$$

However, this function is in a weaker substitute space for L^1, namely $L^{1,\infty}$ (recall Definition 1.6).

Theorem 7.8. Let K be a Calderón-Zygmund kernel and T its associated operator. Then for each $p \in (1, \infty)$, there is a constant $C_{p,n}$ such that

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} B \|f\|_{L^p(\mathbb{R}^n)} .$$

The main estimate will be derived from the following proposition.

Proposition 7.9. The following estimate holds true

$$[Tf]_{1,\infty} \leq C(n) B \|f\|_{L^1} .$$

One strategy to control this is to split a function f into two components $f = f_1 + f_2$, where $f_1 \in L^\infty \cap L^1$. Then, this component can be dealt with since it is in all of L^p, and we can use -for instance- the L^2 theory. In our case, we need to refine this decomposition and force a bit more structure on f_2.

Lemma 7.10 (*Calderón-Zygmund decomposition*). Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Then, we may write

$$f = g + b,$$

where

(i) $\|g\|_{L^1} \leq \|f\|_{L^1}$

(ii) $\|g\|_{L^\infty} \leq 2^n \lambda$

(iii) $\|b\|_{L^1} \leq 2 \|f\|_{L^1}$

where $\mathcal{B}$ is a collection of ***bad*** cubes which are almost disjoint and range over dyadic cubes. Furthermore,

$$\left| \bigcup_{Q \in \mathcal{B}} Q \right| \leq \lambda^{-1} \|f\|_{L^1} ,$$

and

$$\lambda < |Q|^{-1} \int_Q |f(x)| \, \mathrm{d}x \leq 2^n \lambda, \quad \forall Q \in \mathcal{B}.$$

Proof. To begin with, let f and λ be as in the statement. We introduce the collection of dyadic cubes $\mathcal{B}$. Specifically, for each dyadic number 2^l, $l \in Z$, we consider the grid of cubes whose faces are parallel to coordinate planes of edge length 2^l, and whose vertices are among the grid $\left\{2^l k_1, 2^l k_2, \ldots, 2^l k_n\right\}$, for $k_j \in \mathbb{Z}$.

Start with $l_* \in \mathbb{Z}$, chosen large enough such that

$$⨍_Q |f(x)| \, \mathrm{d}x \leq \lambda,$$

for all dyadic cubes Q of edge-length 2^{l_*}. Hence ***at level*** l_*, all cubes Q are ***good***. Then, subdivide Q into 2^n cubes of edge length 2^{l_*-1}. At this point, we have two possibilities

(i) If $⨍_{Q'} |f(x)| \, \mathrm{d}x > \lambda$ for one of these smaller ones, we stop the process and add Q to $\mathcal{B}$.

(ii) Otherwise, we continue the subdivision process for Q'.

Finally, we set

$$b(x) := \sum_{Q\in\mathcal{B}} \left[f(x) - \left(\fint_Q f(y)\,\mathrm{d}y \right) \right] \mathbb{1}_Q(x), \quad g := f(x)\mathbb{1}_{\left(\bigcup_{Q\in\mathcal{B}}\right)^c}(x) + \sum_{Q\in\mathcal{B}} \left(\fint_Q f(y)\,\mathrm{d}y \right) \mathbb{1}_Q(x).$$

Then $f(x) = g(x) + b(x)$ and it remains to check the statement of the lemma.

First, we prove the L^∞ estimate on g. Let $x \in \mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{B}} Q$. Then there exists a sequence $\{Q_i\}_{i\in I}$ with Q_i of edge length 2^{l_i}, with $l_i \to -\infty$, and such that $x \in \bigcap Q_i$ and with every Q_i being a good cube,

$$\fint_{Q_i} |f(x)|\,\mathrm{d}x \le \lambda.$$

Then, for such x, we have $|f(x)| \le \lambda$ thanks to the Lebesgue differentiation theorem. Further, if $Q \in \mathcal{B}$, then by construction, it arose by subdividing a ***good cube*** $\tilde{Q}$ containing it. Therefore,

$$\lambda < \fint_Q |f(x)|\,\mathrm{d}x \le \frac{2^n}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|\,\mathrm{d}x \le 2^n\lambda. \tag{7.1}$$

Using the right-hand side inequality,

$$|g(x)| \le \lambda \mathbb{1}_{\left(\bigcup_{Q\in\mathcal{B}}\right)^c}(x) + \sup_{Q\in\mathcal{B}} \left(\fint_Q |f(y)|\,\mathrm{d}y \right) \mathbb{1}_{\bigcup_{Q\in\mathcal{B}}}(x) \le \max\{\lambda, 2^n\lambda\} \le 2^n\lambda.$$

We now prove the volume bound on $\mathcal{B}$. To this end, notice that the left-hand side inequality in (7.1) yields

$$|Q| < \lambda^{-1} \int_Q |f(x)|\,\mathrm{d}x. \tag{7.2}$$

Since the $Q \in \mathcal{B}$ are almost everywhere disjoint (this is, their intersection is a set of zero measure), this implies that

$$\left| \bigcup_{Q\in\mathcal{B}} Q \right| \le \sum_{Q\in\mathcal{B}} |B| \le \lambda^{-1} \sum_{Q\in\mathcal{B}} \int_Q |f(x)|\,\mathrm{d}x \le \lambda^{-1} \|f\|_{L^1}.$$

Finally, the L^1 estimates on g and b follow easily from their definition,

$$\|g\|_{L^1} = \left\| f\mathbb{1}_{\left(\bigcup_{Q\in\mathcal{B}} Q\right)^c} \right\|_{L^1} + \sum_{Q\in\mathcal{B}} \|f\mathbb{1}_Q\|_{L^1} = \|f\|_{L^1},$$

as well as

$$\|b\|_{L^1} = \sum_{Q\in\mathcal{B}} \int_Q \left| f(x) - \fint_Q f(y)\,\mathrm{d}y \right| \mathrm{d}x \le \sum_{Q\in\mathcal{B}} \int_Q |f(x)|\,\mathrm{d}x + |Q| \fint_Q |f(y)|\,\mathrm{d}y = 2\|f\|_{L^1}.$$

□

Before we move onto the proof of Proposition 7.9, we will need one more additional result.

Lemma 7.11. Let $f \in L^1(\mathbb{R}^n)$ and b and $\mathcal{B}$ as in the Calderón-Zygmund decomposition (Lemma 7.10). For each $Q \in \mathcal{B}$, define Q^* to be the $(2\sqrt{n})$-dilate of Q. Then

$$\left| \bigcup_{Q \in \mathcal{B}} Q^* \right| \leq \frac{C_n}{\lambda} \|f\|_{L^1}, \quad \text{and } \|Tb\|_{L^1\left(\mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{B}} Q^*\right)} \leq \|f\|_{L^1}.$$

Proof. Regarding the volume estimate, recall (7.2), and notice that, since the dilation factor is chosen uniformly,

$$\left| \bigcup_{Q \in \mathcal{B}} Q^* \right| \leq C_n \sum_{Q \in \mathcal{B}} |Q| \leq \frac{C_n}{\lambda} \sum_{Q \in \mathcal{B}} \int_Q |f(x)| \,\mathrm{d}x \leq \frac{C_n}{\lambda} \|f\|_{L^1}.$$

Notice that the dilation factor ensures that for each $Q \in \mathcal{B}$, if $x \in (Q^*)^c$ and $y \in Q$, then for any two $y_1, y_2 \in Q$, we have

$$|x - y| \geq 2\,|y_1 - y_2|.$$

In particular, for any $Q \in \mathcal{B}$, we will consider the choice $y_1 = y$, $y_2 = y_Q$, the center of the cube Q. Write now

$$b_Q = \left[f(x) - \left(\fint_Q f(y) \,\mathrm{d}y \right) \right] \mathbb{1}_Q(x),$$

so that $b = \sum_{Q \in \mathcal{B}} b_Q$, and notice that

$$Tb(x) = \sum_{Q \in \mathcal{B}} b_Q(x) = \sum_{Q \in \mathcal{B}} \int K(x-y) B_Q(y) \,\mathrm{d}y = \sum_{Q \in \mathcal{B}} \int_Q [K(x-y) - K(x - y_Q)] b_Q(y) \,\mathrm{d}y$$

since b_Q averages zero on Q. Now, rewrite the kernel as

$$K(x-y) - K(x-y_q) = K(x-y) - K(x - y - (y - y_Q)).$$

By performing an integration over x and replacing x by $x - y$ for y fixed, the Hörmander condition applies and we reach the estimate

$$\int_{\left(\bigcup_{Q\in\mathcal{B}} Q^*\right)^c} |K(x-y) - K(x-y_Q)| \,\mathrm{d}x \leq \int_{|\tilde{x}| > 2|y - y_Q|} |K(\tilde{x}) - K(\tilde{x} - (y_Q - y))| \,\mathrm{d}\tilde{x} \leq B.$$

Calling $S = \mathbb{R}^n \setminus \bigcup_{Q \in \mathcal{B}} Q^*$, and using Fubini's theorem and the triangle inequality,

$$\begin{aligned}
\|Tb\|_{L^1(S)} &= \int_{\left(\bigcup_{Q\in\mathcal{B}} Q^*\right)^c} |Tf_2(x)| \,\mathrm{d}x \\
&= \int_{\left(\bigcup_{Q\in\mathcal{B}} Q^*\right)^c} \left| \sum_{Q \in \mathcal{B}} [K(x-y) - K(x-y_Q)] f_Q(y) \,\mathrm{d}y \right| \mathrm{d}x \\
&\leq B \sum_{Q \in \mathcal{B}} \int |f_Q(y)| \,\mathrm{d}y \leq B \|f\|_{L^1}.
\end{aligned}$$

□

We now turn to the proof of the weak estimate.

Proof of Proposition 7.9. Notice that is suffices to show that

$$|\{|Tf| > \lambda\}| \leq \frac{C(n)}{\lambda} B \|f\|_{L^1},$$

which, replacing T by $B^{-1}T$ and λ by $B^{-1}\lambda$, we reduce to the case $B = 1$,

$$|\{|Tf| > \lambda\}| \leq C(n)\lambda^{-1} \|f\|_{L^1}.$$

We now make a few observations. First of all, after applying the Calderón-Zygmund decomposition, we estimate $A_\lambda = |\{|Tf| > \lambda\}|$. To this end, we exploit the decomposition $Tf = Tg + Tb$. Indeed, notice that if $|Tf| > \lambda$, then either $|Tg| \geq \lambda/2$ or $Tb \geq \lambda/2$. Hence, if we assign volumes A_1 and A_2 to these sets, respectively, we have

$$A_\lambda \leq A_1 + A_2, \quad A_1 = |V_1| = \left|\left\{|Tg| \geq \frac{\lambda}{2}\right\}\right|, \quad A_2 = |V_2| = \left|\left\{|Tb| \geq \frac{\lambda}{2}\right\}\right|$$

We now bound each of these separately. For A_1, we take advantage of the fact that $g \in L^1 \cap L^\infty$, which implies an L^2-estimate,

$$\int_{\mathbb{R}^n} |g(x)|^2 \, \mathrm{d}x \leq C(n) \|g\|_{L^\infty} \int_{\mathbb{R}^n} |g(x)| \, \mathrm{d}x = \lambda C(n) \|f\|_{L^1},$$

where, actually, $C(n) = 1 + 2^n$. Then by the L^2-boundedness of T, we have

$$|A_1| \leq \frac{4}{\lambda^2} \|Tg\|_{L^2}^2 \leq \frac{4}{\lambda^2} \|g\|_{L^1}^2 \leq \frac{4C(n)}{\lambda} \|f\|_{L^1}.$$

Where, from the preceding, we estimated the L^2 norm of g in terms of its L^1 and L^∞ norms.

We now bound A_2. The key idea for this part is to split and estimate V_2 appropriately. Indeed, write

$$V_2 \subset \left(\bigcup_{Q\in\mathcal{B}} Q^*\right) \cup \left(V_2 \setminus \bigcup_{Q\in\mathcal{B}} Q^*\right).$$

Lemma 7.11 provides a bound for the volume of the first set, while for the second one, we may proceed as follows:

$$\left|\left\{x \in \left(\bigcup_{Q\in\mathcal{B}} Q^*\right)^c : |Tf_2(x)| \geq \frac{\lambda}{2}\right\}\right| \leq \frac{2}{\lambda} \int_{\left(\bigcup_{Q\in\mathcal{B}}\right)^c} |Tb(x)| \, \mathrm{d}x$$

by the Chebyshev inequality, and Lemma 7.11 again provides a bound for this part. Hence

$$A_2 \leq \frac{C_n}{\lambda} \|f\|_{L^1}.$$

Putting the two estimates for A_1 and A_2 together completes the proof. □

At this point, we are ready to show that Calderón-Zygmund operators are indeed bounded in L^p.

Proof of Theorem 7.8. By the Marcinkiewicz interpolation result, we can interpolate for $p \in (1, 2]$, finding $C_{p,n}$ for which

$$\|Tf\|_{L^p} \leq C_{p,n} B \|f\|_{L^p}.$$

Finally, if $p \geq 2$, then we argue by duality and using Fubini's theorem,

$$\begin{aligned} \|Tf\|_{L^p} &= \sup_{\|g\|_{L^{p'}} \leq 1} \left| \int Tf(x)\overline{g}(x)\,\mathrm{d}x \right| = \sup_{\|g\|_{L^{p'}}} \left| \int f(x)\overline{T^*g(x)}\,\mathrm{d}x \right| \\ &\leq \sup_{\|g\|_{L^{p'}} \leq 1} \|f\|_{L^p} \|T^*g\|_{L^{p'}} \leq C_{p',n} B \|f\|_{L^p} \|g\|_{L^{p'}}, \end{aligned}$$

where T^* is the Calderón-Zygmund operator with kernel $K^*(x) = \overline{K(-x)}$, and therefore

$$\|Tf\|_{L^p} \leq C_{p',n} B \|f\|_{L^p}. \qquad \square$$

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