Functional Analysis II

Lecture notes for a one semester course taught by Professor Marc Burger.

Contents include Banach algebras, C^*-algebras, Guelfand theory and Fourier analysis on locally compact abelian groups.

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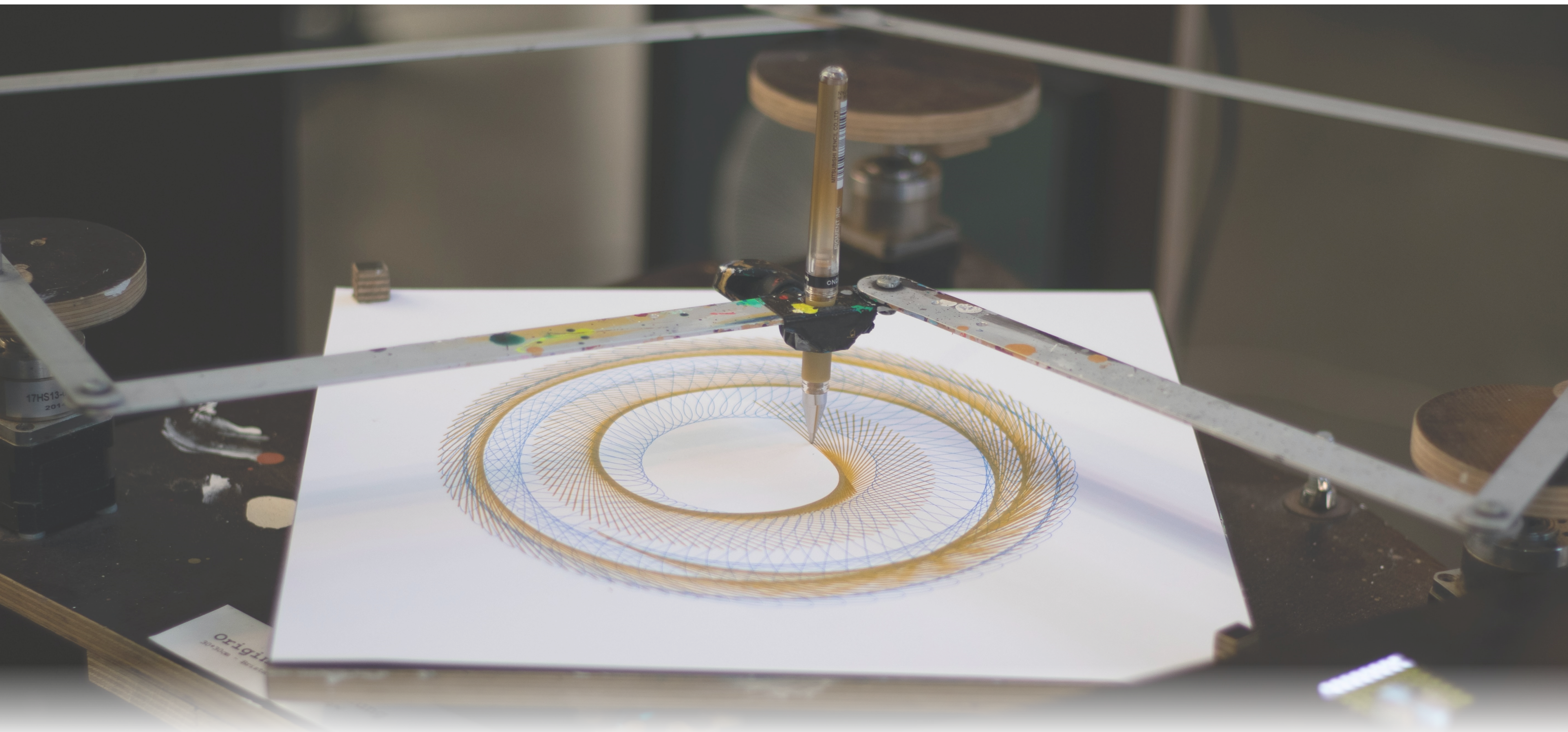

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Introduction

These are lecture notes for a course on functional analysis taught by Professor Marc Burger during spring semester 2022. The course begins by introducing $\mathbb{C}$-algebras, Banach algebras and C^*-algebras and the spectrum of elements of an algebra. Then it moves onto characters and the Guelfand spectrum, endowing it with the Guelfand topology and establishing a relationship between the spectrum of an element and its Guelfand transform. This relationship is then explored in the special case of C^*-algebras. The course continues with spectral theory and some applications before moving onto Fourier analysis on groups.

We deeply thank Professor Marc Burger not only for his great lectures, but also for his incredible help in writing these notes.

Contents

1 Banach algebras

The material of this chapter is mainly taken from [Ka] 1.1 and [GeRaSh] I.1.

We begin by recalling some basic notions.

Definition 1.1 ($\mathbb{C}$-*algebra*). A $\mathbb{C}$-algebra, or simply an algebra, is a $\mathbb{C}$-vector space A endowed with a bilinear map

$$\begin{aligned} A \times A &\longrightarrow A \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

called product (or multiplication) and such that

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall x, y, z \in A.$$

We can introduce richer structure on an algebra by requesting some additional properties.

Definition 1.2. An algebra A is called unital if there exists $e \in A$, $e \neq 0$ such that

$$e \cdot x = x \cdot e = x \quad \forall x \in A.$$

It is called abelian (or commutative) if

$$x \cdot y = y \cdot x \quad \forall x, y \in A.$$

Definition 1.3. An ideal in A is a $\mathbb{C}$-vector subspace I of A such that

$$x \cdot I \subset I,\ I \cdot x \subset I \quad \forall x \in A.$$

It is then a standard fact that the product on A descends to a well defined product on A/I.

We can canonically embed any $\mathbb{C}$-algebra A in a unital algebra, which we will usually denote by A_I, by means of the following construction. Define $A_I := A \times \mathbb{C}$ and introduce the operation

$$(x, \lambda) \cdot (y, \mu) = (xy + \lambda y + \mu x, \lambda\mu).$$

This defines an algebra structure on A_I with unit $e := (0, 1)$ into which A embeds via the map $x \mapsto (x, 0)$ as an ideal.

Now we move on to the main content of the course. In fact, the following definition is central to it.

Definition 1.4. A Banach algebra is a $\mathbb{C}$-algebra A endowed with a norm $\|\cdot\|$ for which A is a Banach space, and which satisfies

$$\|x \cdot y\| \leq \|x\|\|y\| \quad \forall x, y \in A.$$

Similarly to how we constructed a unital algebra from an algebra A, we can define an "extension" of a norm on A into A_I, namely by defining

$$\|(x, \lambda)\| = \|x\| + |\lambda|\,.$$

Then A_I becomes a unital Banach algebra.

Exercise

Let A be a unital $\mathbb{C}$-algebra with a Banach space norm for which the product in A is continuous. Then there is an equivalent norm $\|\cdot\|_*$ for A in which it remains a unital Banach algebra and $\|e\|_* = 1$.

The following definition will appear in several examples further in the course.

Definition 1.5. If a Banach algebra A admits a map $x \mapsto x^*$ with the following properties:

(i) $(x^*)^* = x$,

(ii) $(x + y)^* = x^* + y^*$,

(iii) $(\alpha x)^* = \bar{\alpha} x^*$,

(iv) $(xy)^* = y^* x^*$,

(v) $\|x^*\| = \|x\|$,

for every $x, y \in A$ and $\alpha \in \mathbb{C}$, we say A is an involutive Banach algebra and the map $x \mapsto x^*$ is the involution on A. If the involution satisfies the additional condition:

$$\|x^* \cdot x\| = \|x^*\|\|x\| \quad \forall x \in A,$$

then A is called a C^*-algebra.

Exercise

If e is a unit in an involutive Banach algebra then $e^* = e$.

There are ***three main classes*** of Banach algebras. These can roughly be described as algebras of functions with pointwise multiplication, algebras of operators with composition of operators, and group algebras with convolution product. We now give examples of Banach algebras in each of these three classes.

Example 1.6 (*Banach Algebras*).

(i) **Commutative C^*-algebra**. Let X be a locally compact Hausdorff topological space.

The $\mathbb{C}$-vector space

$$C^b(X) := \{f\colon X \longrightarrow \mathbb{C} :\ f \text{ is continuous and bounded}\}$$

is a $\mathbb{C}$-algebra for pointwise multiplication of functions. The norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)|$$

makes it a commutative unital algebra. By defining $f^*(x) := \overline{f(x)}$, the map $f \mapsto f^*$ is an involution, and in fact it is a C^*-algebra. Recall that $f\colon X \longrightarrow \mathbb{C}$ vanishes at infinity if for every $\varepsilon > 0$ there exists $K \subset X$ compact such that $|f(x)| < \varepsilon$ for every $x \in X \setminus K$. The space

$$C_0(X) = \{f\colon X \longrightarrow \mathbb{C} :\ f \text{ is continuous and vanishes at infinity}\},$$

endowed with the norm $\|\cdot\|_\infty$ is a commutative C^*-algebra; it is unital if and only if X is compact. In fact, as a consequence of Guelfand's theory of commutative Banach algebras, we will see that any commutative C^*-algebra is isomorphic to $C_0(X)$ for some appropriate X.

(ii) Let $X \subset \mathbb{C}$ be a compact set. Define

$$A(X) = \{f\colon X \longrightarrow \mathbb{C} :\ f \text{ is continuous and holomorphic in the interior of } X\}.$$

Then A, together with $\|\cdot\|_\infty$, is a unital Banach subalgebra of $C(X)$. Observe that if $\operatorname{Int} X \neq \emptyset$, then $f^*(\xi) = \overline{f(\xi)}$ is not an involution since f^* is not holomorphic in $\operatorname{Int} X$ in general.

(iii) Let $a < b$ and $n \in \mathbb{N}$. Let

$$C^n([a,b]) = \{f\colon [a,b] \longrightarrow \mathbb{C} :\ f \text{ is } n\text{-times continuously differentiable}\}$$

with pointwise multiplication and the norm given by

$$\|f\| := \sum_{k=0}^{n} \frac{1}{k!}\|f^{(k)}\|_\infty,$$

where $f^{(k)}$ denotes the k-th derivative of f. Then it follows that $C^n([a,b])$ is a Banach space. In addition,

$$\begin{aligned}\|f\cdot g\| = \sum_{k=0}^{n} \frac{1}{k!}\|(f\cdot g)^{(k)}\|_\infty = \sum_{k=0}^{n}\|\sum_{j=0}^{k} \frac{1}{j!(k-j)!} f^{(j)} g^{(k-j)}\|_\infty \leq \\ \leq \sum_{l=0}^{n}\sum_{s=0}^{n} \frac{1}{l!}\|f^{(l)}\|_\infty \frac{1}{s!}\|g^{(s)}\|_\infty = \|f\|\|g\|.\end{aligned}$$

Thus $C^n([a,b])$ is a commutative unital Banach algebra. It is a nontrivial fact that $C^\infty([a,b])$ does not admit any Banach algebra norm (see Example 3.19).

(iv) **Volterra Algebra**. Let λ be the Lebesgue measure on $\mathbb{R}$ restricted to $[0,1]$ and $L^1([0,1])$ the space of measurable, absolutely integrable functions on $[0,1]$ with the L^1-norm. It is a classical fact that $L^1([0,1])$ is a Banach space. Using Fubini's theorem one can show that $f * g(x)$, defined by

$$f * g(x) = \int_0^x f(x-t)g(t)\,\mathrm{d}t,$$

exists and belongs to $L^1([0,1])$ for λ-a.e. $x \in [0,1]$. The operation given by convolution is obviously bilinear, and by substituting $t \mapsto x - t$ we can clearly see that it is commutative. It is also easy to check that

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1,$$

thus making $L^1([0,1])$ a commutative Banach algebra. It is an interesting exercise to show that it has no unit.

(v) Let B be a Banach space with norm $\|\cdot\|$ and

$$\mathcal{L}(B) := \{T\colon B \longrightarrow B : \ T \text{ is linear and continuous}\}.$$

Then $\mathcal{L}(B)$ is a $\mathbb{C}$-algebra for the composition. It is a fact from functional analysis that $\mathcal{L}(B)$ is characterized as the space of linear maps $T : B \longrightarrow B$ for which

$$\|T\| := \sup_{\|x\| \leq 1} \|T(x)\| < \infty.$$

Furthermore, $\|T_1 T_2\| \leq \|T_1\|\|T_2\|$ and $\mathcal{L}(B)$ is a unital Banach algebra. However, it is not commutative unless $\dim B = 1$.

(vi) Let now $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Recall that the adjoint T^* of $T \in \mathcal{L}(\mathcal{H})$ is uniquely defined by

$$\langle T^* v, w \rangle = \langle v, Tw \rangle \quad \forall v, w \in \mathcal{H}.$$

Then $(T^*)^* = T$ and it is easy to verify the first four properties that an involutive Banach algebra satisfies. As for the last two, if $v \in \mathcal{H}$, we have

$$\|Tv\|^2 = \langle Tv, Tv \rangle = \langle T^*Tv, v \rangle \leq \|T^*Tv\|\|v\| \leq \|T^*T\|\|v\|^2,$$

which implies $\|T\|^2 \leq \|T^*T\|$. Together with $\|T^*T\| \leq \|T^*\|\|T\|$, thus implies that $\|T\| \leq \|T^*\|$, and hence

$$\|T\| \leq \|T^*\| \leq \|(T^*)^*\| = \|T\|.$$

Therefore $\|T^*\| = \|T\|$.

For the sixth property we come back to $\|T\|^2 \leq \|T^*T\|$ and deduce from the earlier statements that

$$\|T^*T\| = \|T^*\|\|T\|.$$

Hence $\mathcal{L}(\mathcal{H})$ with the operation $T \mapsto T^*$ is a unital C^*-algebra.

In particular, Guelfand's structure theorem for commutative C^*-algebras (see chapter 4) can be applied to commutative sub-C^*-algebras of $\mathcal{L}(\mathcal{H})$ leading to spectral theorems for unitary, self adjoint, or more generally normal operators.

(vii) Let Γ be a group, for example $SL(n, \mathbb{Z})$ or simply $\{1\}$. Then

$$\ell^1(\Gamma) = \left\{ f\colon \Gamma \longrightarrow \mathbb{C} : \sum_{\gamma \in \Gamma} |f(\gamma)| < \infty \right\}$$

is a Banach space when endowed with the usual ℓ_1 norm

$$\|f\|_1 = \sum_{\gamma \in \Gamma} |f(\gamma)| .$$

Recall that $\mathbb{C}[\Gamma]$ is the $\mathbb{C}$-vector space with basis $\{\delta_\gamma : \gamma \in \Gamma\}$ where

$$\delta_\gamma(\mu) = \begin{cases} 1, & \mu = \gamma, \\ 0, & \mu \neq \gamma. \end{cases}$$

That is, $\mathbb{C}[\Gamma]$ consists of formal linear combinations of elements of Γ. Define the following product on the basis elements by

$$\delta_\gamma \cdot \delta_\eta := \delta_{\gamma\eta}$$

and extend by linearity to $\mathbb{C}[\Gamma]$. Then it is easy to verify this is a $\mathbb{C}$-algebra. Furthermore, $\mathbb{C}[\Gamma]$ can be seen as a subset of $\ell^1(\Gamma)$ and the above product extends to $\ell^1(\Gamma)$. For $f, g \in \ell^1(\Gamma)$, we extend it by

$$f * g(\gamma) = \sum_{\eta \in \Gamma} f(\gamma\eta) g(\eta^{-1})$$

and call it the convolution product. Let us estimate its norm:

$$\|f * g\|_1 = \sum_{\gamma \in \Gamma} \left| \sum_{\eta \in \Gamma} f(\eta\gamma) g(\eta^{-1}) \right| \leq \sum_{\eta \in \Gamma} \left(\sum_{\gamma \in \Gamma} |f(\gamma\eta)| \right) \left| g(\eta^{-1}) \right| =$$
$$= \left(\sum_{\gamma \in \Gamma} |f(\gamma)| \right) \left(\sum_{\eta \in \Gamma} |g(\eta)| \right) = \|f\|_1 \|g\|_1 .$$

One can then define an involution as $f^*(\gamma) := \overline{f(\gamma^{-1})}$. With this, $\ell^1(\Gamma)$ is an involutive Banach algebra.

2 Spectrum of a Banach algebra element

The material of this chapter is mainly taken from [Ka] 1.2.

Definition 2.1. Let A be a $\mathbb{C}$-algebra with unit $e \in A$. An element $x \in A$ is called invertible if there exists a $y \in A$ with $xy = yx = e$. Then y is uniquely determined and is denoted x^{-1}.

The set of invertible elements $G(A) = \{x \in A : x \text{ is invertible}\}$ is a group with natural identity element e.

Definition 2.2 (*Spectrum*). Let A be a unital algebra. For $x \in A$, define

$$\operatorname{Sp}_A(x) := \{\lambda \in \mathbb{C} : \ x - \lambda e \text{ is not invertible}\},$$

as the ***spectrum*** of x. Its complement, $\rho_A(x) = \mathbb{C} \setminus \operatorname{Sp}_A(x)$, is called the ***resolvent set***.

If A has no unit, then for $x \in A$ we define $\operatorname{Sp}_A(x) = \operatorname{Sp}_{A_I}(x)$, and $\rho_A(x) = \rho_{A_I}(x)$.

Example 2.3. Let $A = \operatorname{End}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space. For $T \in A$, $\operatorname{Sp}_A(T)$ is the set of roots of the characteristic polynomial of T. Notice that $T - \lambda\,\mathrm{id}$ is not invertible if and only if $\ker(T - \lambda\,\mathrm{id}) \neq \{0\}$, which is equivalent to $c_T(\lambda) = \det(T - \lambda\,\mathrm{id}) = 0$. The Fundamental Theorem of Algebra implies that $\operatorname{Sp}_A(T) \neq \emptyset$.

One of the main results of this section is that for any Banach algebra, the spectrum of an element is a nonempty set. In fact, we will prove that the ***spectral radius formula***, which given x, provides the radius of the smallest closed disc centered at $0 \in \mathbb{C}$ that contains the spectrum of x. We will now present a few lemmas that are going to be useful for proving the spectral radius formula.

Definition 2.4. For $x \in A$, define $r_A(x) = \inf\left\{\|x^n\|^{1/n} : n \geq 1\right\}$.

Since $\|x^n\| \leq \|x\|^n$, we get $\|x^n\|^{1/n} \leq \|x\|$ and therefore $r_A(x) \leq \|x\|$. In addition, we have $r_A(\lambda x) = |\lambda|\, r_A(x)$ for every $\lambda \in \mathbb{C}$ and every $x \in A$. This key remark will allow us to say something about $r_A(x)$ in the case $\|x\| < 1$, i.e., when x is inside the unit ball. Now, for the following lemma we use the (not generally appreciated) convention that the natural numbers include 0, and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Lemma 2.5. Let $f\colon \mathbb{N}^* \longrightarrow \mathbb{R}_{\geq 0}$ with $f(n+m) \leq f(n)f(m)$ for all $n, m \geq 1$. Then

$$\lim_{n\to\infty} f(n)^{1/n} = \inf\{f(n)^{1/n} : n \geq 1\}.$$

Proof. Let $r = \inf\left\{f(n)^{1/n} :\ n \geq 1\right\}$. Let $\varepsilon > 0$ and pick $k \geq 1$ such that $f(k)^{1/k} < r + \varepsilon$. Let $n \geq k$ and write $n = ak + b$, for $a \geq 1$ and $0 \leq b \leq k-1$. Then

$$f(n) = f(ak+b) \leq f(k)^a f(1)^b,$$

and hence

$$\begin{aligned} f(n)^{1/n} &\leq f(k)^{a/n} f(1)^{b/n} = \left(f(k)^{1/k}\right)^{ak/n} f(1)^{b/n} = \left(f(k)^{1/k}\right)^{1-b/n} f(1)^{b/n} \\ &= \left(f(k)^{1/k}\right)^{1-b/n} f(1)^{b/n} \leq (r+\varepsilon)^{(1-b/n)} f(1)^{b/n}. \end{aligned}$$

Therefore we can bound the lim sup in the following way,

$$\limsup_{n\to\infty} f(n)^{1/n} \leq r + \varepsilon,$$

and therefore, since lim inf is greater or equal that inf, which is precisely r, we get

$$\limsup_{n\to\infty} f(n)^{1/n} \leq r \leq \liminf_{n\to\infty} f(n)^{1/n}.$$

□

From here we obtain the following corollary.

Corollary 2.6. For every $x \in A$, we have

$$\lim_{n\to\infty} \|x^n\|^{1/n} = r_A(x).$$

To prove this, we can simply use the previous lemma with $f(n) = \|x^n\|$.

The next results introduce criteria for the invertibility of an element $x \in A$ and they give us information about the topological properties of $G(A)$. This will help us prove that $\mathrm{Sp}_A(x)$ is a closed set. The first of these lemmas, along with the remark we made after the definition of $r_A(x)$, allows us to check that every element x in the unit ball makes $e - x$ invertible, giving us an explicit expression for its inverse.

Lemma 2.7. Let A be a Banach algebra with identity $e \in A$ and $x \in A$ with $r_A(x) < 1$. Then $e - x$ is invertible and

$$(e-x)^{-1} = e + \sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} x^n,$$

where the series is absolutely convergent.

Recall that in a general Banach space we say the series $\sum_{n=1}^{\infty} a_n$, $a_n \in A$, converges if the sequence of partial sums converges in the norm topology. We say that it converges absolutely if $\sum_{n=1}^{\infty} \|a_n\|$ converges. To prove that absolute convergence implies convergence it is necessary to prove that the sequence of partial sums is Cauchy by means of the triangle inequality. Similarly to the real case, absolute convergence implies convergence.

Proof. The idea here is to compare the series to a geometric one. Pick a number $r_A(x) < q < 1$. By the previous corollary, take $N \geq 1$ with $\|x^n\|^{1/n} \leq q$ for every n greater than N. This means that $\|x^n\| \leq q^n$. At the same time this implies that the series

$$\sum_{n=1}^{\infty} \|x^n\|$$

converges since eventually it compares to a geometric series on q. Hence, $\sum_{n\geq 1} x^n$ converges in A. Let

$$S_n = e + \sum_{k=1}^{n} x^k, \quad y := \lim_{n\to\infty} S_n.$$

Then $S_n \cdot x = S_{n+1} - e$. In a similar way, $x \cdot S_n = S_{n+1} - e$. By letting n tend to infinity, it follows that $yx = y - e$ from the first equality and $xy = y - e$ from the second one. Therefore $e = y(e - x)$ and $e = (e - x)y$. □

Remark. If $\|x\| < 1$ then $r_A(x) \leq \|x\| < 1$ and $e - x$ is invertible. This means that if we take an open ball of radius 1, it is entirely contained in $G(A)$, the group of invertible elements.

Lemma 2.8. Let A be a Banach algebra with an identity element $e \in A$.

(i) For all x, y in $G(A)$, if $\|y - x\| \leq \frac{1}{2}\|x^{-1}\|^{-1}$, then $\|y^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2\|y - x\|$. This means that in a neighborhood of x, the map $y \mapsto y^{-1}$ is Lipschitz continuous

(ii) $G(A)$ is open in A and if $x \in A$ is such that $\|e - x\| < 1$ then $x \in G(A)$.

Proof. For the first part, write $y^{-1} - x^{-1} = y^{-1}(x - y)x^{-1}$. Since A is a Banach algebra, we obtain

$$\|y^{-1} - x^{-1}\| \leq \|y^{-1}\|\|x - y\|\|x^{-1}\|. \tag{2.1}$$

In particular, by the inverse triangle inequality together with the hypotheses, this means that

$$\|y^{-1}\| - \|x^{-1}\| \leq \|y^{-1}\|\|x - y\|\|x^{-1}\| \leq \frac{1}{2}\|y^{-1}\|.$$

Therefore $\frac{1}{2}\|y^{-1}\| \leq \|x^{-1}\|$, and inserting this into (2.1) shows (i).

For the second part, notice that if $\|e - x\| < 1$, then $r_A(e - x) < 1$ and $x = e - (e - x) \in G(A)$. Now if $x \in G(A)$ and $y \in A$ with $\|y - x\| < \|x^{-1}\|^{-1}$, we can estimate

$$\|x^{-1}y - e\| = \|x^{-1}(y - x)\| \leq \|x^{-1}\|\|y - x\| < 1,$$

which implies $x^{-1}y \in G(A)$, and hence $y \in G(A)$. □

This shows that the inverse map is a homeomorphism of $G(A)$ into itself. The following theorem is one of the most fundamental results in the theory of Banach algebras. It uses some basic facts from the theory of holomorphic functions, as well as the uniform boundedness theorem from functional analysis.

Theorem 2.9. Let A be a Banach algebra, $x \in A$. Then the spectrum $\text{Sp}_A(x)$ is a nonempty compact subset of $\mathbb{C}$ and

$$r_A(x) = \max\left\{|\lambda| : \lambda \in \text{Sp}_A(x)\right\}.$$

Recall that if A is unital, $\text{Sp}_A(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible}\}$ and if A is not unital, then we define $\text{Sp}_A(x) = \text{Sp}_{A_I}(x)$, since otherwise it could be argued that no element is invertible and the first expression would make no sense. Before proving the theorem, let us introduce some notation.

Definition 2.10 (*Spectral radius*). For $x \in A$, we denote by

$$\|x\|_{\text{Sp}} := \sup\left\{|\lambda| : \lambda \in \text{Sp}_A(x)\right\}$$

the ***spectral radius*** of x.

Theorem 2.9 tells us that $\|x\|_{\text{Sp}}$ could be defined as a maximum, but we do not know of its existence beforehand. The theorem states that for every element $x \in A$, its spectrum is a nonempty compact set that is sharply contained in a ball of radius $\|x\|_{\text{Sp}}$, and there exists a point λ in it which has $\lambda = \|x\|_{\text{Sp}}$. An important result that follows from theorem 2.9 is the following corollary.

Corollary 2.11 (*Guelfand-Mazur*). If A is a unital Banach algebra such that every nonzero element is invertible, then $A \cong \mathbb{C}$.

This result is of key importance since in the future we will take the quotient of A by some ideal in order to establish a similar situation as that of the corollary, and thus prove that the quotient is isomorphic to $\mathbb{C}$. Additionally, it is foundational to much of Guelfand's theory and can be seen as a generalisation of Frobenius' theorem, which states that a finite dimensional $\mathbb{C}$-algebra in which every nonzero element is invertible is itself $\mathbb{C}$.

Now, before giving an argument for this, let us continue with the proof of theorem 2.9.

Proof of theorem 2.9. We may first assume that A is unital. If not, we adjoin the identity and in view of the definition of the spectrum of an element, this does not affect the result.

Step 1. First, let us show that $\text{Sp}_A(x)$ is contained within a disk of some big enough radius (namely $r_A(x)$). Write

$$\lambda e - x = \lambda\Big(e - \frac{x}{\lambda}\Big)$$

so if $r_A\left(\frac{x}{\lambda}\right) < 1$, then $e - \frac{x}{\lambda}$ is invertible by Lemma 2.7 and so is $\lambda e - x$. Now, by checking that $r_A(x/\lambda) = r_A(x)/\,|\lambda|$ from the definition, we obtain that if $|\lambda| > r_A(x)$, then $\lambda \notin \text{Sp}_A(x)$. By the contrapositive of this last statement, it follows that $\|x\|_{\text{Sp}} \leq r_A(x)$.

Step 2. Observe that $\text{Sp}_A(x)$ is a closed subset of $\mathbb{C}$; indeed Sp_A is the inverse image of $A \backslash G(A)$ under the continuous map

$$\begin{aligned} \mathbb{C} &\longrightarrow A \\ \lambda &\longmapsto \lambda e - x \end{aligned}$$

but $G(A)$ is the group of invertible elements, which is open in A by Lemma 2.8. Since the spectrum is a closed, bounded subset, it is compact.

Step 3. Let us see that $\mathrm{Sp}_A(x)$ is nonempty. Begin by setting $\ell \in A^*$ to be a continuous linear form from A to $\mathbb{C}$. Now, for every λ in the resolvent set, $\rho_A(x) = \mathbb{C} \setminus \mathrm{Sp}_A(x)$, define the function $f(\lambda) = \ell\Big((\lambda e - x)^{-1}\Big)$. Let $\mu \in \rho_A(x)$ and compute

$$\begin{aligned} f(\lambda) - f(\mu) &= \ell\left((\lambda e - x)^{-1} - (\mu e - x)^{-1}\right) \\ &= \ell\bigg(\frac{(\mu e - x) - (\lambda e - x)}{(\lambda e - x)(\mu e - x)}\bigg) = (\mu - \lambda)\ell\bigg(\frac{1}{(\lambda e - x)(\mu e - x)}\bigg). \end{aligned}$$

Therefore if $\lambda \neq \mu$,

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = \ell\bigg(\frac{1}{(\lambda e - x)(\mu e - x)}\bigg).$$

The map

$$\begin{aligned} \rho_A(x) &\longrightarrow G(A) \\ \mu &\longmapsto (\mu e - x)^{-1} \end{aligned}$$

is continuous by Lemma 2.8. Hence

$$\lim_{\mu \to \lambda} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -\ell\bigg(\frac{1}{(\lambda e - x)^2}\bigg).$$

Therefore, $f\colon \rho_A(x) \longrightarrow \mathbb{C}$ is holomorphic. The idea now is to assume the spectrum is empty. This means that f is defined over all of $\mathbb{C}$. However, by the definition of f, when $|\lambda|$ increases, f vanishes. Since f is entire, we reach a contradiction and hence $\mathrm{Sp}_A(x)$ cannot be an empty set. In order to make this rigorous, we are going to derive a power series expansion for $e - \frac{x}{\lambda}$ so that we can investigate the behavior of f at ∞. We know that if $|\lambda| > r_A(x)$ then $r_A\left(\frac{x}{\lambda}\right) < 1$ and we have an absolutely convergent series expansion by Lemma 2.7:

$$\left(e - \frac{x}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{x}{\lambda}\right)^n.$$

Hence

$$(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} x^n.$$

Therefore,

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} \ell(x^n),$$

which is allowed since the series for $(\lambda e - x)^{-1}$ is absolutely convergent for $|\lambda| > r_A(x)$ and ℓ is continuous as well as linear. Now we proceed with the argument. Assume that $\mathrm{Sp}_A(x) = \emptyset$. Then f is entire and if $|\lambda| \geq 2\|x\|$, then

$$\begin{aligned} |f(\lambda)| = \left| \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} \ell(x^n) \right| &\leq |\lambda|^{-1} \sum_{n=0}^{\infty} (2\|x\|)^{-n} \|\ell\| \|x\|^n = \\ &= |\lambda|^{-1} \|\ell\| \sum_{n=0}^{\infty} \frac{1}{2^n} = 2\,|\lambda|^{-1}\,\|\ell\|. \end{aligned}$$

In particular, by Liouville's theorem, f must be constant (since it is holomorphic on $\mathbb{C}$ and bounded) and since $2\,|\lambda|^{-1}\,\|\ell\| \to 0$ as $|\lambda| \to \infty$ this constant must be 0. But this means that for all $\ell \in A^*$

$$\ell((\lambda e - x)^{-1}) = 0$$

hence $(\lambda e - x)^{-1} = 0$ which is impossible. Therefore $\operatorname{Sp}_A(x) \neq \emptyset$.

Step 4. We proceed to show that $r_A(x) \leq \|x\|_{\mathrm{Sp}}$. Recall that in the first step of the proof we established that $\|x\|_{\mathrm{Sp}} \leq r_A(x)$. Now define, for $|\xi| < 1/r_A(x)$,

$$g(\xi) = f(1/\xi) = \sum_{n=0}^{\infty} \xi^{n+1} \ell(x^n).$$

Notice that g is holomorphic in $|\xi| < 1/r_A(x)$. Also, f is holomorphic on $\mathbb{C} \setminus \operatorname{Sp}_A(x) \supset \mathbb{C} \setminus \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|_{\mathrm{Sp}}\}$, and thus g is holomorphic in the disk $\left\{\xi \in \mathbb{C} : |\xi| < \frac{1}{\|x\|_{\mathrm{Sp}}}\right\}$. As a consequence, the Taylor expansion of g at 0 converges for all $|\xi| < 1/\|x\|_{\mathrm{Sp}}$. In particular, for every such ξ,

$$\sup_{n\geq 1} \left|\xi^{n+1}\ell(x^n)\right| < \infty \quad \forall \ell \in A^*$$

by the expression for the Taylor expansion of g and its convergence. By the uniform boundedness principle, $\sup_{n\geq 1}\|\xi^{n+1}x^n\| < \infty$. Therefore there exists a constant $c(\xi)$ for which $\|\xi^n x^n\| \leq c(\xi)$, and $\|x^n\| \leq c(\xi)\,|\xi|^{-n}$. Hence

$$\|x^n\|^{1/n} \leq c(\xi)^{1/n}\,|\xi|^{-1}.$$

If we take the limit when $n \to \infty$, we get $r_A(x) \leq |\xi|^{-1}$, for all $|\xi| < 1/\|x\|_{\mathrm{Sp}}$. This proves $r_A(x) \leq \|x\|_{\mathrm{Sp}}$, which together with the inequality in step 1 proves the theorem. □

Now we give a proof for corollary 2.11.

Proof of corollary 2.11. Let $x \in A$. We know that $\operatorname{Sp}_A(x)$ is nonempty, and hence there exists $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is not invertible. By the hypotheses, $\lambda e - x = 0$. This determines the following map. Let us denote by $\lambda(x)$ the unique complex number with $\lambda(x)e - x = 0$. Now it is an easy exercise to see that $\lambda\colon A \longrightarrow \mathbb{C}$ is the desired isomorphism. Continuity follows directly from the fact that it is a vector space isomorphism, making A a finite dimensional Banach algebra. Since $|\lambda(\cdot)|$ is a norm, it is continuous. This isomorphism is unique: $\mathrm{id}_{\mathbb{C}}$ is the only $\mathbb{C}$-algebra automorphism of $\mathbb{C}$. □

2.1 Holomorphic functional calculus

This short section will be dedicated to some natural developments which are part of a more comprehensive treatment of the theory of Banach algebras. We will however not use any of these results later on.

Let $x \in A$ and $f\colon \mathbb{C} \longrightarrow \mathbb{C}$. We ask the question of how to extend f to A, i.e., how to define $f(x)$. If f is entire, then we know that

$$\sum \frac{f^{(n)}(0)}{n!} z^n$$

is absolutely convergent: we can therefore define $f(x)$ as this series evaluated in x [1]. Additionally, this works well with the exponential. On the other hand, if f is only holomorphic in an open set $U \supset \operatorname{Sp}_A x$, then the idea is to choose a path that goes around the spectrum once and try to give a meaning to the integral over C of $\frac{f(\lambda)}{\lambda e - x}$,

$$\frac{1}{2\pi i} \oint_C \frac{f(\lambda)}{\lambda e - x}\, \mathrm{d}x.$$

[1] Convergence in A is inherited by absolute convergence in the complex plane.

The way to realise this idea is to take this integral over a compact set, and therefore the function inside –as a function of λ– is uniformly continuous, resulting in the Riemann sum converging in the Banach algebra. This limit is uniquely characterized by

$$\varphi(y) = \frac{1}{2\pi i}\oint_C f(\lambda)\varphi\big((\lambda e - x)^{-1}\big)\,\mathrm{d}\lambda, \quad \varphi \in A^*.$$

Let us state this in greater detail.

Let A be a unital Banach algebra. For every polynomial

$$P(X) = aX^n + \ldots + a_0 \in \mathbb{C}[X]$$

and every element $x \in A$, the evaluation of P in x is well defined as an element of A,

$$P(x) = ax^n + \ldots + a_0 \in A.$$

In fact, this can be extended to a ring of holomorphic functions in the following way. Let $x \in A$, and $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function defined over an open set U containing $\mathrm{Sp}_A(x)$. Let $C \subset U$ be a simple closed curve enclosing $\mathrm{Sp}_A(x)$. For every $\varphi \in A^*$, define

$$F(\varphi) := \frac{1}{2\pi i}\int_C f(\lambda)\varphi\big((\lambda - x)^{-1}\big)\,\mathrm{d}\lambda.$$

Proposition 2.12. $F \in A^{**}$ is in fact represented by a vector $v \in A$, that is,

$$F(\varphi) = \varphi(v) \quad \forall \varphi \in A^*.$$

We will denote this vector by $f(x) \in A$; symbolically, $f(x)$ is given by

$$f(x) = \frac{1}{2\pi i}\int_C f(\lambda)(\lambda - x)^{-1}\,\mathrm{d}\lambda.$$

Proposition 2.13. The map $f \mapsto f(x) \in A$ is a homomorphism of the algebra $\mathcal{A}\big(\mathrm{Sp}_A(x)\big)$ of all functions that are holomorphic in a neighborhood of $\mathrm{Sp}_A(x)$ into A, which sends the constant function to the identity $1 \in A$ and the function $\lambda \mapsto \lambda$ to x.

Theorem 2.14 (*Spectral mapping theorem*). Let A be a unital Banach algebra, $x \in A$, and f a holomorphic function on a neighborhood of $\mathrm{Sp}_A(x)$. Then

$$\mathrm{Sp}_A(f(x)) = f\big(\mathrm{Sp}_A(x)\big)\,.$$

Moreover, if g is holomorphic on a neighborhood of $f\big(\mathrm{Sp}_A(x)\big)$, then

$$g \circ f(x) = g(f(x))\,.$$

3 The Guelfand representation

The material of this chapter is mainly taken from [Ta] I.3. An alternative source is e.g. [RaVa], chapter 2.

Our goal now is, given a commutative algebra A, to construct its Guelfand spectrum

$$\widehat{A} = \{\varphi\colon A \longrightarrow \mathbb{C} : \ \varphi \text{ is a homomorphism}\}$$

and endow it with a topology. We will later study its properties, along with its relation to the original algebra. For this, we first need to develop some machinery regarding ideals in commutative Banach algebras. For this reason, throughout this whole chapter we will consider A to be a commutative Banach algebra. The following concept captures a key idea that is necessary for the discussion.

Definition 3.1. An ideal $I \subset A$ is ***regular*** if there exists $u \in A$ such that $ux - x \in I$ for all $x \in A$.

Notice that in case I is regular and proper, then A/I is unital, in other words, I is regular if either the quotient A/I admits $[u]$ as an identity element or $A/I = (0)$. Also, if A is unital, then e is an identity for every proper ideal, meaning that if A is unital, then every ideal is regular. For this reason it is interesting to find regular ideals in non-unital algebras. It resembles finding ideals of A that are unital, in an algebra that is non-unital. Now we will give a few examples.

Example 3.2 (*Regular ideals*).

(i) Let $\chi\colon A \longrightarrow \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism such that $\chi \neq 0$ and $\chi(A) = \mathbb{C}$. Then $\ker \chi$ is a regular ideal of A. This is proven through noticing that $1 = \chi(u)$ for some $u \in A$.

(ii) Any ideal in a unital Banach algebra is regular. This is the trivial example.

(iii) An important example that will appear throughout the rest of the chapter is the following. Let X be a locally compact Hausdorff space. Recall that $C_0(X)$ consists of all the continuous functions from X to $\mathbb{C}$ that vanish at infinity. Define

$$I(E) = \{f \in C_0(X) : \ f|_E = 0\}\,,$$

for E a closed subset of X. Then $I(E)$ is regular if and only if E is compact. In order to prove this, we need the following lemma.

Lemma 3.3 (*Urysohn's lemma*). Let X be a locally compact, Hausdorff space. Suppose $K \subset V \subset X$, where K is compact and V is open. Then there exists $f : X \longrightarrow [0,1]$ such that $f|_K = 1$, $\operatorname{supp} f \subset V$ and f has compact support.

Now we argue. For the direct implication, we need $u \in C_0(X)$ such that $u|_E = 1$ (in order to achieve $(uf - f)|_E = 0$). For this reason we require E to be compact. For the converse, the map u from above is given by Urysohn's lemma.

The following proposition tells us that u cannot be too close to the ideal I, since otherwise it would get absorbed, preventing it from being regular.

Proposition 3.4. Let $I \subset A$ be a proper, regular ideal, and let $u \in A$ be a unit modulo I. Then

$$\|u - x\| \geq 1 \quad \forall x \in I.$$

Proof. Assume that there exists a $x \in I$ such that $\|u - x\| < 1$. By Lemma 2.7 the sum $s = \sum_{n=1}^{\infty} (u - x)^n$ is absolutely convergent. Now,

$$u - x = s - s(u - x) = s - su + sx.$$

Note that $s - su \in I$ by definition of u, $sx \in I$ by the definition of an ideal, and $x \in I$ by assumption. Therefore $u \in I$ and thus $A/I = 0$. Hence, $I = A$, which contradicts the assumption that I is proper. □

Definition 3.5 (*Maximal ideal*). An ideal $I \subset A$ is maximal if any ideal $I \subseteq M \subseteq A$ is either I or A.

We recall here the definition of ***maximal ideal*** in order to introduce the following corollary, in which we see that regular ideals can be widened to their closure given that the distance from elements of $\overline{I}$ does not increase that of I.

Corollary 3.6. The closure of any proper, regular ideal in A is a proper and regular ideal. In particular, any maximal regular ideal is closed.

Proof. Let $I \subset A$ be a regular and proper ideal. Then $\bar{I}$ is an ideal because of the continuity of the algebraic operations. $\overline{I}$ is regular because $I \subset \overline{I}$. It is also proper: for u a unit modulo I, it holds that $\|u - x\| \geq 1$ for every $x \in I$, and thus the same holds for $x \in \bar{I}$. □

The following lemma gives us a sense of how a proper regular ideal also satisfies the property that it is contained in some maximal ideal, same as ideals in algebras. Following a similar vein to what we did before, we can widen a regular ideal since u will still be the unit for A/J whenever J contains I and u is not in it.

Lemma 3.7. Every regular and proper ideal is contained inside a maximal proper and regular ideal.

Proof. Let $I \subset A$ be a regular and proper ideal in A. Define

$$\mathcal{L} = \{J \subset A : \ I \subset J,\ u \notin J \ \text{ and } J \text{ is an ideal}\}\,.$$

The idea now is to order $\mathcal{L}$ by the inclusion and check that it satisfies the hypotheses of Zorn's lemma. We endow $\mathcal{L}$ with the order defined by the inclusion. Let $\mathcal{K}$ be a totally ordered subset of $\mathcal{L}$. Then define

$$L = \bigcup_{K \in \mathcal{K}} K,$$

and notice that L is an ideal and $u \notin L$, since otherwise it would be in some $K \in \mathcal{K}$. Therefore L is maximal in $\mathcal{K}$. Thus, by Zorn's lemma, there exists J which is maximal in $\mathcal{L}$. It is proper as well, since $J \in \mathcal{L}$ implies $u \notin J$. □

Corollary 3.8. Let A be a unital commutative Banach algebra. If $x \in A$ is not invertible then x is contained in a maximal proper ideal.

Proof. Let $e \in A$ be the identity and consider the proper ideal $A \cdot x$. Then $e \notin A \cdot x$, so $A \cdot x$ is regular. Hence by Lemma 3.7 there exists M a maximal regular proper ideal such that $A \cdot x \subset M$. Thus, $x \in M$. □

The next proposition shows that the Banach algebra structure on A induces a Banach algebra structure on quotients by a closed ideal.

Proposition 3.9. Let A be a Banach algebra and $I \subset A$ a closed ideal. Then A/I is also a Banach algebra with the quotient algebra structure.

Proof. Let $\pi\colon A \to A/I$ be the quotient map. We have that:

- A/I is an algebra with multiplication $(x+I)(y+I) = xy + I$.
- A/I is a Banach space with norm $\|x+I\|_I \coloneqq \inf\{\|x+u\| : \ u \in I\}$.

We have to show that $\|\pi(x)\pi(y)\|_I \le \|\pi(x)\|_I \|\pi(y)\|_I$ for every $x, y \in A$. Let $\varepsilon > 0$ and $u, v \in I$ such that

$$\begin{aligned} \|x+u\| &\le \|\pi(x)\|_I + \varepsilon. \\ \|y+v\| &\le \|\pi(y)\|_I + \varepsilon. \end{aligned}$$

Then we can estimate

$$\begin{aligned} \|\pi(x)\pi(y)\|_I &= \|\pi(x+u)(y+v)\|_I \le \|(x+u)(y+v)\| \le \\ &\le \|x+u\|\|y+v\| \le (\|\pi(x)\|_I + \varepsilon)(\|\pi(y)\|_I + \varepsilon) \end{aligned}$$

Since ε was arbitrary, we conclude. □

We turn to an application of Urysohn's lemma to determine all the closed ideals in the particularly important example of $C_0(X)$, which we will need later in the course.

Proposition 3.10. Let X be a locally compact Hausdorff space and define

$$I(E) \coloneqq \{f \in C_0(X) : \ f|_E = 0\} \quad \forall E \subset X.$$

Then
$$\begin{array}{ccc} \{E \subset X : E \text{ is closed}\} & \longrightarrow & \{I \subset C_0(X) : I \text{ is a closed ideal}\} \\ E & \longmapsto & I(E) \end{array}$$
is a bijection. Moreover, $I(E)$ is maximal if and only if $|E| = 1$.

Proof. We divide the proof into several steps.

Step 1. If $E \subset X$ is arbitrary, then $I(E)$ is closed since $I(E) = I(\bar{E})$ by continuity of functions in $C_0(X)$. Secondly, by Urysohn's lemma, $I(E)$ is proper whenever $E \neq \emptyset$, and if E_1, E_2 are distinct closed sets in X, then $I(E_1) \neq I(E_2)$ so the map from the statement is indeed an injection.

Step 2. Now we prove it is surjective. Given a proper, closed ideal $I \subset C_0(X)$, define $E := \{x \in X : f(x) = 0\ \forall f \in I\}$. By definition of E, we have the inclusion $E \subset X$, closedness of E, and the inclusion $I \subset I(E)$. To prove the converse inclusion (and hence the equality), we proceed in two steps:

(i) Firstly, we prove that every $g \in C_c^0(X)$ (i.e. continuous function with compact support) with $\operatorname{supp} g \cap E = \emptyset$ belongs to I.

(ii) Secondly, we prove that the set $J = \left\{g \in C_c^0(X) : \operatorname{supp} g \cap E = \emptyset\right\}$ is dense in $I(E)$. Since $I \subset I(E)$ is closed, we get the equality by density of I in $I(E)$.

Let us turn to the first claim, for which we need the following fact:

($*$) Let $C \subset X$ be a compact set, $C \cap E = \emptyset$. Then for all $x \in C$ there exists $h_x \in I$ with $h_x(x) \neq 0$ and $(h_x)^2 \in I$. Since C is a compact, there exists $F \subset C$ such that $h = \sum_{x \in F} (h_x)^2 \in I$ and h is strictly positive on C.

Let J be as in (ii) and $g \in J$: then by ($*$) there exists $h \in I$ such that $h(y) > 0$ for all $y \in \operatorname{supp}(g)$. Define
$$f(x) := \begin{cases} 0 & x \notin \operatorname{supp} g \\ \frac{g(x)}{h(x)} & x \in \operatorname{supp} g. \end{cases}$$
It is easy to verify that f is continuous and $f \in C_c^0(X)$. Since $h \in I$ and $g = fh \in I$, it must happen that $g \in I$, concluding the proof of (i).

For the second claim, let $\varepsilon > 0$ and $f \in I(E)$. Define $C := \{x \in X : |f(x)| \geq \varepsilon\}$. Then C is compact, and $C \cap E = \emptyset$. Therefore, by Urysohn's lemma there exists a map $h : X \longrightarrow [0,1]$ that is continuous and compactly supported, and such that $h|_C = 1$ and $\operatorname{supp} h \subset X \setminus E$. Define $g = fh \in J$ and $\|f - g\|_\infty \leq \varepsilon$. This concludes the proof. □

3.1 Characters and the Guelfand spectrum

We now introduce the construction of the Guelfand spectrum and its topology. To this end, we introduce its elements and the relation it bears with ideals in A, giving meaning to the discussion about regular ideals that precedes this section.

Definition 3.11 (*Characters*). Let A be a commutative Banach algebra. A character of A is a $\mathbb{C}$-algebra homomorphism $\chi \colon A \longrightarrow \mathbb{C}$ that is not identically zero.

Notice that a character χ is always surjective, since $\chi(A) \subset \mathbb{C}$ is a nonzero vector subspace. Now we define and associate the Guelfand spectrum to a commutative Banach algebra.

Definition 3.12 (*Guelfand spectrum*). Let A be a commutative Banach algebra. We denote by $\widehat{A}$ the set of characters of A and call it the Guelfand spectrum of A.

We first establish the relation between $\widehat{A}$ and $\widehat{A_I}$.

Remark. Every $\varphi \in \widehat{A}$ has a unique extension $\tilde{\varphi}$ to A_I given by

$$\tilde{\varphi}(x, \lambda) = \varphi(x) + \lambda.$$

Let $\tilde{\widehat{A}} = \left\{\tilde{\varphi} : \varphi \in \widehat{A}\right\} \subset \widehat{A}_I$. If $\varphi_\infty : A_I \longrightarrow \mathbb{C}$ denotes the character given by $\varphi_\infty(x, \lambda) = \lambda$, then $\widehat{A}_I = \tilde{\widehat{A}} \cup \{\varphi_\infty\}$. In this heuristic we will, whenever convenient, identify $\widehat{A}$ with $\tilde{\widehat{A}}$ and drop the latter notation.

The following proposition is a little miracle in the theory of abelian Banach algebras.

Proposition 3.13. For every $\varphi \in \widehat{A}$ we have

$$|\varphi(x)| \leq \|x\|_{\text{Sp}} \quad \forall x \in A.$$

In particular, φ is a bounded linear functional with $\|\varphi\| \leq 1$ in general, and $\|\varphi\| = 1$ if A is unital and the norm of the unit is 1.

Proof. Because of the previous remark, we may assume A is unital. If $|\lambda| > \|x\|_{\text{Sp}}$, then $x - \lambda e$ is invertible and since $\varphi(e) = 1$, we conclude $\varphi(x) \neq \lambda$. Hence

$$|\varphi(x)| \leq \|x\|_{\text{Sp}} \leq \|x\| \quad \forall x.$$

If A is unital, then $\varphi(e) = 1$ and if additionally $\|e\| = 1$, we thus have $\|\varphi\| = 1$. ☐

Example 3.14. It is an exercise to check that if $A = L^1([0, 1])$ is the Volterra algebra, then

$$\|f\|_{\text{Sp}} = 0 \quad \forall f \in A.$$

Hence $\widehat{A} = \emptyset$.

The following result gives the relationship between characters and maximal regular ideals, announced at the beginning of this chapter. It is for the proof of this theorem that we developed the previous machinery about ideals inside commutative Banach algebras. The theorem allows us to identify characters in the spectrum of A and certain ideals in A.

Theorem 3.15. For a commutative Banach algebra the mapping

$$\varphi \longmapsto \ker \varphi$$

establishes a bijection between $\widehat{A}$ and the set $\operatorname{Max} A$ of maximal regular proper ideals in A.

Proof. We will divide the proof into three steps.

Step 1. For $\varphi \in \widehat{A}$, $\ker\varphi$ is an ideal and it is maximal since it is of codimension 1 in A. Moreover, it is regular by Example 3.2.

Step 2. If $\ker\varphi_1 = \ker\varphi_2 =: I$ let $u \in A$ be an identity element modulo I. Since $I + \mathbb{C}u = A$, let $x = x_0 + \lambda u$. Then since $\varphi_1(u) = \varphi_2(u) = 1$,

$$\varphi_1(x) = \lambda\varphi_1(u) = \lambda = \lambda\varphi_2(u) = \varphi_2(x).$$

Step 3. Let $I \in \operatorname{Max} A$ and $u \in A$ an identity modulo I. By Corollary 3.6 I is closed and hence by Proposition 3.9 A/I is a Banach algebra with respect to the quotient algebra structure. By assumption A/I is unital with neutral element $u + I$. We claim that every $x + I$ with $x \notin I$ is invertible. If not, then $(x + I)A/I := J$ is a proper ideal of A/I and hence its inverse image under the canonical projection $\pi\colon A \to A/I$ is again an ideal satisfying

$$I \subsetneq \pi^{-1}(J) \subsetneq A,$$

contradicting the maximality of I. Hence by the Guelfand-Mazur theorem A/I is isomorphic to $\mathbb{C}$, which leads to a character $\chi \in \widehat{A}$ with $\ker\chi = I$. □

For the following example, recall Proposition 3.10.

Example 3.16. Let X be locally compact Hausdorff. Then for $x \in X$, the evaluation map $\varphi \mapsto \varphi(x)$ defines a character of $C_0(X)$ which leads to a bijection $X \longrightarrow \widehat{C_0(X)}$.

Definition 3.17. Let A be an algebra. Define the radical of A as

$$\operatorname{Rad} A := \bigcap \{M : M \in \operatorname{Max} A\}.$$

We say A is semisimple if $\operatorname{Rad} A = (0)$.

Notice that if every maximal ideal that appears in the definition above were regular, then we would get

$$\operatorname{Rad} A = \bigcap \left\{\ker\varphi : \varphi \in \widehat{A}\right\},$$

meaning that if A is a commutative Banach algebra, then this holds.

The automatic continuity of characters given by Proposition 3.13 has some consequences for $\mathbb{C}$-algebra homomorphisms with values in semisimple commutative Banach algebras. Let us explore them.

Corollary 3.18. Let $\varphi\colon A \longrightarrow B$ be a $\mathbb{C}$-algebra homomorphism with A, B commutative Banach algebras and B semisimple. Then φ is continuous.

Before proving this result we may recall the ***Closed Graph Theorem***.

Theorem (*Closed Graph*). Let $T\colon E \longrightarrow F$ be a linear map of Banach spaces. Then the following are equivalent:

(i) T is continuous.

(ii) Graph $T \subset E \times F$ is closed.

(iii) If $x_n \to 0$ in E and $Tx_n \to y$ in F, then $y = 0$.

Now we turn to the proof of the corollary.

Proof. Let $\{x_n\}_{n\geq 1}$ be a sequence in A with $x_n \to 0$ and $\varphi(x_n) \to b$ in B. Let $\chi \in \widehat{B}$. Then $\chi \circ \varphi \in \widehat{A} \cup \{0\}$ and both χ and $\chi \circ \varphi$ are continuous by Proposition 3.13. Therefore

$$\chi(b) = \lim_{n\to\infty} \chi(\varphi(x_n)) = \lim_{n\to\infty} (\chi \circ \varphi)(x_n) = (\chi \circ \varphi)(0) = 0,$$

and $b = 0$ because B is semisimple. □

We have seen in Example 1.6 that for every $n \geq 0$, $C^n([0,1])$ admits a Banach algebra norm. One can use Corollary 3.18 to show the following.

Example 3.19. The space $C^\infty([0,1])$ does not admit any Banach algebra norm. The idea is the following: Assume $\|\cdot\|$ is a norm on $C^\infty([0,1])$. The algebra $C([0,1])$ is semisimple and the inclusion $C^\infty([0,1]) \hookrightarrow C([0,1])$ is hence continuous. Thus, there exists $c > 0$ with $\|f\|_\infty \leq c\|f\|$ for every $f \in C^\infty([0,1])$. By means of this inequality and the Closed Graph Theorem one can show that the derivative $D\colon f \mapsto f'$ is continuous. A contradiction is then reached by considering the functions $t \mapsto e^{\alpha t}$ for every $\alpha \in \mathbb{C}$.

We further expose another example the importance of which will become clear in later chapters.

Example 3.20. Consider $A = \ell^1(\Gamma)$, where Γ is an abelian group; for example one may take a finite group, or the integers, etc. Let us compute $\widehat{A}$. If $\chi \in \widehat{\ell^1(\Gamma)}$, then in particular $\chi \in \ell^1(\Gamma)^* = \ell^\infty(\Gamma)$. That is, there exists a unique $H \in \ell^\infty(\Gamma)$ with

$$\chi(f) = \sum_{\gamma\in\Gamma} f(\gamma)H(\gamma) \quad \forall f \in \ell^1(\Gamma).$$

Evaluating χ on δ_γ gives $\chi(\delta_\gamma) = H(\gamma)$ and from the multiplicativity of χ we get

$$H(\gamma\eta) = \chi(\delta_{\gamma\eta}) = \chi(\delta_\gamma * \delta_\eta) = \chi(\delta_\gamma)\chi(\delta_\eta) = H(\gamma)H(\eta).$$

Also $H(e) = \chi(\delta_e) = 1$ and from $\|H\|_\infty = \|\chi\| = 1$ we deduce $|H(\gamma)| = 1$ for every $\gamma \in \Gamma$. Letting

$$\mathbb{T} := \{\xi \in \mathbb{C} : |\xi| = 1\}$$

we have obtained the identification $\widehat{\ell^1(\Gamma)} \cong \mathrm{Hom}(\Gamma, \mathbb{T})$. We will later see that $\ell^1(\Gamma)$ is semisimple, but this fact lies deeper.

3.2 Guelfand topology

Now we move to a crucial point, namely that the Guelfand spectrum can be equipped with a natural topology. One approach to do this, would be to observe that $\widehat{A}$ is contained in the unit ball of the dual A^* of A and restrict the weak*-topology to $\widehat{A}$. We choose to pursue a different, self-contained approach and only make use of Tychonov's theorem.

We will now introduce a topology on the Guelfand spectrum of a Banach algebra A. Declare a subset $W \subset \widehat{A}$ to be open if it is the empty set or if for every $\varphi_0 \in W$ there exists $\varepsilon > 0$ and $a_1, ..., a_n \in A$ such that

$$\mathcal{U}(\varphi_0; a_1, ..., a_n; \varepsilon) := \left\{ \varphi \in \widehat{A} : \ |\varphi(a_i) - \varphi_0(a_i)| < \varepsilon, \ 1 \le i \le n \right\}$$

is completely contained within W. In other words, the set

$$\left\{ \mathcal{U}(\varphi_0; a_1, ..., a_n; \varepsilon), \ \varphi_0 \in \widehat{A}, n \ge 1, \{a_1, ..., a_n\} \subset A, \varepsilon > 0 \right\}$$

is the basis of open sets of a topology on $\widehat{A}$, called the Guelfand topology.

Exercise

Show that the Guelfand topology is the weakest topology on $\widehat{A}$ with respect to which all the functions

$$\begin{aligned} f_x : \widehat{A} &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \varphi(x) \end{aligned}$$

for $x \in A$ are continuous.

Here is a first fundamental fact:

Theorem 3.21.

(i) $\widehat{A}$ is locally compact Hausdorff.

(ii) $\widehat{A}$ is compact if A is unital.

(iii) $\widehat{A}_I$ is the one point compactification of $\widehat{A}$.

By this point it may be useful for the reader to revisit the one-point compactification of a topological space. To this end, we dedicate the first appendix. Now recall ***Tychonov's theorem***: given a arbitrary family $\{X_i : i \in I\}$ of compact topological spaces, the product space $\prod_{i \in I} X_i$ is always compact when equipped with the product topology.

Proof of theorem 3.21. We first show that $\widehat{A}$ is compact if A is unital. Under this assumption, for every $x \in A$, it holds that $|\varphi(x)| \le \|x\|$, for all $\varphi \in \widehat{A}$. Let $\overline{D}(0, r) := \{z \in \mathbb{C} : \ |z| \le r\}$, and consider the map

$$\begin{aligned} \Phi : \widehat{A} &\longrightarrow \textstyle\prod_{x \in A} \overline{D}(0, \|x\|) \\ \varphi &\longmapsto \quad (\varphi(x))_{x \in A}. \end{aligned}$$

When equipped with the product topology, one verifies easily that $\Phi : \widehat{A} \to \Phi(\widehat{A})$ is a homeomorphism onto its image. Next we indicate how to show that $\Phi(\widehat{A})$ is closed; together with Tychonov's theorem this implies that $\widehat{A}$ is compact.

Let $(\lambda_x)_{x \in A} \in \prod_{x \in A} D(0, \|x\|)$ be in the closure of $\Phi(\widehat{A})$. Let $\varepsilon > 0, x, y \in A, \alpha, \beta \in \mathbb{C}$ be given. Furthermore, let $\varphi \in \widehat{A}$ be such that:

$$|\varphi(a) - \lambda_a| < \varepsilon \quad \forall a \in \{x, y, x \cdot y, \alpha x + \beta y\}.$$

Then using appropriate triangle inequalities:

$$|\alpha \lambda_x + \beta \lambda_y - \lambda_{\alpha x + \beta y}| \le \varepsilon(|\alpha| + |\beta| + 1)$$

$$|\lambda_{xy} - \lambda_x \cdot \lambda_y| \leq \varepsilon(1 + \|y\| + \|x\|)$$

which by arbitrariness of ε implies that $\lambda\colon A \to \mathbb{C}$ is a $\mathbb{C}$-algebra homomorphism. Since $\varphi(e) = 1\ \forall\varphi \in \widehat{A}$, we get that $\lambda_e = 1$ and $\lambda \in \Phi(\widehat{A})$.

Now to tackle the general case we drop the hypothesis that A is unital. As previously explained, we may consider $\widehat{A}$ as a subset of $\widehat{A}_I = \widehat{A} \cup \{\varphi_\infty\}$. One verifies then that the Guelfand topology on $\widehat{A}$ coincides with the one induced by $\widehat{A}_I$. The statement about the one point compactification is left as an exercise. □

Exercise

Let X be a locally compact Hausdorff space. We had a setwise bijection $X \longrightarrow \widehat{C_0(X)}$. Now with the Guelfand topology on $\widehat{C_0(X)}$ this bijection is a homeomorphism.

The following definition will be of key relevance throughout the rest of the course, and we will identify it in certain examples as an important, well-known operator: the ***Fourier transform***.

Definition 3.22. Let $x \in A$ and define

$$\begin{aligned} \widehat{x}\colon \widehat{A} &\longrightarrow \mathbb{C} \\ \chi &\longmapsto \chi(x). \end{aligned}$$

The map $x \mapsto \widehat{x}$ is the ***Guelfand transform***.

We have then the following result.

Theorem 3.23. If A is an abelian Banach algebra, the Guelfand transform

$$\begin{aligned} A &\longrightarrow C_0(\widehat{A}) \\ x &\longmapsto \widehat{x} \end{aligned}$$

is a homomorphism from A into the abelian C^*-algebra $C_0(\widehat{A})$ of all the continuous functions on $\widehat{A}$ vanishing at infinity. If A is unital, then $\widehat{A}$ is compact and $\mathrm{Sp}_A(x) = \widehat{x}(\widehat{A})$, and if A is not unital, then $\mathrm{Sp}_A(x) = \widehat{x}(\widehat{A}) \cup \{0\}$. In any case $\|\widehat{x}\|_\infty = \|x\|_{\mathrm{Sp}}$ for every $x \in A$.

Proof. If A is unital, the first assertion is clear. If it is not, embed $A \hookrightarrow A_I$ so that every $x \in A$ defines a continuous function on $\widehat{A}_I = \widehat{A} \cup \{\varphi_\infty\}$ satisfying $\widehat{x}(\varphi_\infty) = \varphi_\infty(x) = 0$. Since $\widehat{A} \cup \{\varphi_\infty\}$ is the one-point compactification of $\widehat{A}$, this implies that $\widehat{x}|_{\widehat{A}}$ vanishes at infinity: indeed, $\widehat{x}$ is continuous at φ_∞ and vanishes, which implies that $\forall\varepsilon > 0$ there exists $K \subset \widehat{A}$ such that on $\left(\widehat{A} \setminus K\right) \cup \{\varphi_\infty\}$, $|\widehat{x}| < \varepsilon$.

Now for the other assertions, assume A unital. If $\lambda \in \mathrm{Sp}_A(x)$ then $x - \lambda e$ is not invertible and hence, by Corollary 3.8, it is contained in some maximal regular ideal I. By Theorem 3.15, there is $\varphi \in \widehat{A}$ with $\ker\varphi = I$, so $\varphi(x - \lambda e) = 0$ or equivalently $\varphi(x) = \lambda$. This shows $\mathrm{Sp}_A(x) \subseteq \widehat{x}(\widehat{A})$. Conversely if $\lambda = \varphi(x)$ then $\varphi(x - \lambda e) = 0$ and hence $x - \lambda e$ is not invertible, because φ is a $\mathbb{C}$-algebra-homomorphism. This shows $\widehat{x}(\widehat{A}) \subseteq \mathrm{Sp}_A(x)$.

If A is not unital, embed again $A \hookrightarrow A_I$ and get that for all $x \in A$,

$$\mathrm{Sp}_A(x) = \widehat{x}(\widehat{A} \cup \{\varphi_\infty\}) = \widehat{x}(\widehat{A}) \cup \{0\}\,. \qquad \square$$

4 The Guelfand isomorphism

The material of this chapter is mainly taken from [Ta] I.4 and [Ka] 2.1, 2.2 and 2.3.

The motivation for this section is our need to develop tools to deal with $\mathcal{L}(\mathcal{H})$, the space of linear functionals on a Hilbert space. Recall that an involution is the key object that defines involutive Banach algebras and C^*-algebras. It is a map $x \mapsto x^*$ satisfying a series of important properties. Namely, it is a $\mathbb{C}$-antilinear map, $(x^*)^*$ gives back the original element x, it reverses products, $(xy)^* = y^*x^*$, and it preserves the norm, i.e. $\|x^*\| = \|x\|$. If, in addition, it satisfies $\|xx^*\| = \|x\|\|x^*\|$, then we are dealing with a C^*-algebra.

The main goal of this chapter is to show that for abelian C^*-algebras, the Guelfand transform is actually an isomorphism. First, we shall establish some basic facts about C^*-algebras without assuming that they are abelian. Throughout the entire chapter we will assume that A is an involutive Banach algebra.

Definition 4.1. Let A be an involutive Banach algebra, then $x \in A$ is ***self-adjoint*** if $x = x^*$, it is ***normal*** if it commutes with x^*, and if A has a unit e, then x is ***unitary*** whenever $xx^* = x^*x = e$.

This leads to some easy observations. First of all, elements that are self-adjoint or unitary have to be normal at the same time. More importantly, every $x \in A$ can be written as $x_1 + ix_2$, where x_1, x_2 are self-adjoint, namely

$$x_1 = \frac{x + x^*}{2}, \quad x_2 = \frac{x - x^*}{2i},$$

and this decomposition is unique. Now we explore some fundamental properties of C^*-algebras.

Proposition 4.2. Let A be a C^*-algebra and $x \in A$ normal. Then,

$$\|x\| = \|x\|_{\mathrm{Sp}}.$$

Before proving this proposition, let us give an example operator for which the statement does not hold.

Example 4.3. Take $\mathcal{H} = \mathbb{C}^2$, $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, with $\|T\|_{\mathrm{Sp}} = 1$. Note however that

$$\|T\|^2 = \sup_{\|z\| \leq 1} \langle Tz, Tz \rangle = \sup_{\|z\| \leq 1} \langle T^*Tz, z \rangle$$

is equal to the largest eigenvalue of T^*T, which is $3+2\sqrt{2}>1$. Hence, by Proposition 4.2, T cannot be normal.

We now move on to the proof.

Proof. First, notice that for every $y \in A$, $\|y\|^2 = \|y^*y\|$. Now, for $n \geq 1$, let x be normal in A, and notice that under such assumption,

$$\|x^{2n}\|^2 = \|(x^{2n})^* x^{2n}\| = \|(x^*x)^{2n}\| = \|(x^*x)^n (x^*x)^n\|.$$

Observe that $y = (x^*x)^n$ is self-adjoint, so $\|(x^*x)^n(x^*x)^n\| = \|(x^*x)^n\|^2$. That is, we have reached

$$\|x^{2n}\| = \|(x^*x)^n\|.$$

Applying this we obtain the following

$$\begin{aligned}\|x^{2^n}\| &= \|(x^*x)^{2^{n-1}}\| = \|(x^*x)^{2^{n-2}}(x^*x)^{2^{n-2}}\| \\ &= \|(x^*x)^{2^{n-2}}\|^{2^1} = \|x^{2^{n-1}}\|^{2^1} = \cdots = \|x^*x\|^{2^{n-1}} = \|x\|^{2^n},\end{aligned}$$

and by Theorem 2.9 we end up with

$$\|x\| = \|x^{2^n}\|^{1/2^n} \underset{n\to\infty}{\longrightarrow} \|x\|_{\text{Sp}}.$$ □

We now face another problem. Let A be a non-unital, involutive C^*-algebra. On A_I we can always consider the involution

$$(x,\lambda) \mapsto (x^*, \overline{\lambda}).$$

Using the usual norm $\|(x,\lambda)\| = \|x\| + |\lambda|$ on A_I, we still obtain an involutive Banach algebra but not a C^*-algebra in general,

$$\|(x,\lambda)(x^*,\bar{\lambda})\| = \|xx^* + \bar{\lambda}x + \lambda x^*\| + |\lambda|^2\,,$$

and

$$\|(x,\lambda)^2\| = \|x\|^2 + 2\,|\lambda|\,\|x\| + |\lambda|^2$$

If equality holds, that is, if $\|(x,\lambda)\|^2 = \|(x,\lambda)(x,\lambda)^*\|$, then in particular

$$\begin{aligned}\|x\|^2 + 2\,|\lambda|\,\|x\| + |\lambda|^2 &= \|xx^* + \overline{\lambda}x + \overline{\lambda}x^*\| + |\lambda|^2 \leq \\ &\leq \|xx^*\| + \|\overline{\lambda}x + \lambda x^*\| + |\lambda|^2\,.\end{aligned}$$

This is,

$$2\,|\lambda|\,\|x\| \leq \|\overline{\lambda}x + \lambda x^*\|,$$

so by taking $\lambda = i$, we obtain $2\|x\| \leq \|x^* - x\|$, implying that every self-adjoint element is zero, which results in A being $\{0\}$ (due to the decomposition into self-adjoint elements that we described earlier).

So the natural question is how can we extend a non-unital C^*-algebra to a unital C^*-algebra A_I adequately extending the C^*-norm of A?

Proposition 4.4. Assume A is a non-unital C^*-algebra. Then there is a C^*-algebra norm on A_I extending the given norm on A.

Proof. Let $A_I = A \times \mathbb{C}$ and observe that A is an ideal in A_I. For every $x \in A_I$, consider the left multiplication

$$\begin{aligned} L_x\colon A &\longrightarrow A \\ y &\longmapsto x \cdot y \end{aligned}.$$

We obtain an algebra homomorphism

$$\begin{aligned} A_I &\longrightarrow \mathcal{L}(A) \\ x &\longmapsto L_x. \end{aligned}$$

If $x \in A$, let us compute $\|L_x\|$:

$$\|L_x\| = \sup_{\|y\| \leq 1} \|xy\| \leq \|x\|,$$

and using the fact that $\|xx^*\| = \|x\|^2$, giving $\|x(x^*/\|x\|)\| = \|x\|$, the supremum is attained by $y = x^*/\|x\|$, for which we find $\|L_x\| = \|x\|$ since $\|y\| = 1$.

Notice by this point that this gives us another way to look at the norm of an element $x \in A$. Through this new idea we are able to extend the norm to A_I via the following argument.

Define, for every $x \in A_I$, $N(x) := \|L_x\|$. This clearly satisfies sublinearity and, in fact, all the properties that any Banach algebra norm should fulfill. The only one that is not trivial to see is that $N(x) = 0$ implies $x = 0$. For this, let $x = x' + \lambda \cdot e$ be such that

$$L_x(y) = (x' + \lambda e)y = 0, \quad \forall y \in A.$$

Here, we may assume $\lambda \neq 0$, since for $x \in A$, $\|L_x\| = \|x\|$. This means that $x'y + \lambda y = 0$, so

$$y = \left(-\frac{1}{\lambda}\right) x'y,$$

which defines x' for every $y \in A$. Then

$$y^* = y^* \left(-\frac{1}{\lambda}x'\right)^*.$$

We would get that $-x'/\lambda$ is a left identity on A and $(-x'/\lambda)^*$ a right identity. Therefore

$$\left(-\frac{x'}{\lambda}\right) = \left(-\frac{x'}{\lambda}\right)\left(-\frac{x'}{\lambda}\right)^* = \left(-\frac{x'}{\lambda}\right)^*$$

This implies that both elements are the same and thus it is a right and left identity (i.e. an identity), which is a contradiction to the assumption that A is non-unital. Therefore $N(x) = \|L_x\|$ for $x \in A$ is a Banach algebra norm on A_I.

It remains to verify that N is a C^*-algebra norm on A_I. For all $x \in A_I$, we check that $N(x^*) = N(x)$ easily, and we show that $N(x^*x) = N(x)^2$. For $x \in A_I$ and $a \in A$, given that $xa \in A$,

$$\begin{aligned} \|L_x(a)\|^2 = \|xa\|^2 = \|(xa)^*(xa)\| = \|a^*x^*xa\| \leq \\ \leq \|a^*\|\|x^*xa\| = \|a^*\|\|L_{x^*x}(a)\| \leq \|a\|^2\|L_{x^*x}\| \end{aligned}$$

where in the second step we used that $xa \in A$ and this is a C^*-algebra. Therefore,

$$\|L_x\|^2 \leq \|L_{x^*x}\| \leq \|L_{x^*}\|\|L_x\|,$$

and $N(x)^2 \leq N(xx^*) \leq N(x^*)N(x) = N(x)^2$, showing that $N(x^*x) = N(x)^2$ and thus concluding the proof. □

The next result establishes the natural properties of the spectrum of unitary and self-adjoint elements in a unital C^*-algebra. Proposition 4.4 will be crucial to extend those properties to elements of general C^*-algebras.

Proposition 4.5. Let A be a unital C^*-algebra.

(i) If $u \in A$ is unitary, then $\mathrm{Sp}_A(u) \subset \mathbb{T} := \{\xi \in \mathbb{C} : |\xi| = 1\}$.

(ii) If $h \in A$ is self-adjoint, then $\mathrm{Sp}_A(h) \subset \mathbb{R}$.

Proof. Step 1. We prove the first statement. Let A be a unital C^*-algebra and denote by e a unit in A. Then $ea = ae = a$ implies $a^*e^* = e^*a^* = a^*$, for any $a \in A$, showing that e^* is also an identity and therefore $e = e^*e = e^*$. Furthermore, $\|e\|^2 = \|e^*e\| = \|e\|$ so that $\|e\| = 1$, and unitary elements also have unit norm.

Let $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Then $u - \lambda e = (e - \lambda u^*)u$. However,

$$\|\lambda u^*\| = |\lambda| \, \|u^*\| = |\lambda| < 1,$$

and by Lemma 2.7, $(e - \lambda u^*)$ is invertible, making $u - \lambda e$ invertible.

If, otherwise, $|\lambda| > 1$, then $u - \lambda e = \lambda(u/\lambda - e)$. Since

$$\left\| \frac{u}{\lambda} \right\| = \frac{1}{|\lambda|} < 1,$$

it must happen that $u/\lambda - e$ is invertible. Thus, $\lambda(u/\lambda - e) = u - \lambda e$ is invertible, which concludes the proof of part (i).

Step 2. For the proof of the second part, notice that $\lambda \in \mathbb{C}$ is real if and only if $e^{i\lambda} \in \mathbb{T}$. First, we will show that if $f \colon \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic then for every $x \in A$, $f(x) \in A$ makes sense and $f\big(\mathrm{Sp}_A(x)\big) \subset \mathrm{Sp}_A(f(x))$. Then the result follows by a simple application of this fact: if $h = h^*$, then $\exp(ih)$ is unitary, and it follows that

$$\exp\big(i\mathrm{Sp}_A(h)\big) \subset \mathrm{Sp}_A(\exp(ih)) \subset \mathbb{T}.$$

This immediately implies that $\mathrm{Sp}_A(h) \subset \mathbb{R}$. Therefore we are left to prove the first statement.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion of f at $0 \in \mathbb{C}$. It converges absolutely for all $z \in \mathbb{C}$, and hence,

$$f(x) := \sum_{n=0}^{\infty} a_n x^n, \quad x \in A$$

is well defined by absolute convergence of the series. By x^0 here we mean e. We will write

$$f(x) - f(\lambda)e = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n \lambda^n e = \sum_{n=1}^{\infty} a_n (x^n - \lambda^n e) = \sum_{n=1}^{\infty} (x - \lambda e) p_n(x, \lambda)$$

where the factorization $x^n - \lambda^n e = (x - \lambda e)\big(x^{n-1} + x^{n-2}\lambda + \ldots + x\lambda^{n-2} + \lambda^{n-1}e\big)$ defines $p_n(x,\lambda)$ as the last polynomial in the expression. We can bound it in the following straightforward way,

$$\|p_n(x,\lambda)\| \leq n r^{n-1}, \quad \text{with } r := \max(|x|, |\lambda|).$$

But $f'(\xi) = \sum_{n=0}^{\infty} n a_n \xi^{n-1}$ converges absolutely and uniformly over compact sets, and therefore $\sum_{n=0}^{\infty} n |a_n| r^{n-1}$ converges, making $\sum_{n=0}^{\infty} p_n(x,\lambda)$ absolutely convergent in A. We get

$$f(x) - f(\lambda)e = (x - \lambda e) \sum_{n=0}^{\infty} a_n p_n(x,\lambda) = (x - \lambda e) p(x,\lambda).$$

Hence if $\lambda \in \mathrm{Sp}_A(x)$, $(x - \lambda e)$ is not invertible, making $f(x) - f(\lambda) = (x - \lambda e)p(x, \lambda)$ not invertible. Therefore $f(\lambda) \in \mathrm{Sp}_A(f(x))$. We have shown

$$f\left(\mathrm{Sp}_A(x)\right) \subset \mathrm{Sp}_A(f(x))\,.$$

Finally, let $f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. This defines $\exp x \in A$ for every $x \in A$. Then one shows that if $xy = yx$, then $\exp\,(x + y) = \exp x \exp y$, by means of the binomial theorem. This completes the proof. □

Using Proposition 4.4, we readily obtain the following consequence.

Corollary 4.6. Let A be a C^*-algebra and $h = h^*$ a self-adjoint element. Then $\mathrm{Sp}_A(h) \subset \mathbb{R}$.

Proof. If A is not unital, we have by definition that $\mathrm{Sp}_A(h) = \mathrm{Sp}_{A_I}(x)$. We have seen in Proposition 4.4 that there is a norm on A_I that makes it a C^*-algebra and extends the norm of A. Hence regardless of whether A is unital or not, the spectrum is with respect to a unital C^*-algebra, so the statement follows from Proposition 4.5. □

At this point we are finally ready to introduce the main result we have been building up to. Namely, it states that the Guelfand transform is better than a homomorphism whenever A is an abelian C^*-algebra: it becomes an isomorphism, meaning that A and $C_0(\widehat{A})$ are isomorphic C^*-algebras. More concretely, this gives us a classification of abelian C^*-algebras. Indeed, it tells us that any abelian C^*-algebra is isomorphic to a space of continuous functions vanishing at infinity on a locally compact Hausdorff space. We can therefore start to apply our intuition about continuous functions to the elements of abelian C^*-algebras.

Theorem 4.7. Let A be an abelian C^*-algebra and $\widehat{A}$ its Guelfand spectrum. Then the Guelfand transform $A \longrightarrow C_0(\widehat{A})$ is a C^*-algebra isomorphism.

The proof for this theorem will need the following classical approximation result.

Theorem (*Stone-Weierstrass*). Let X be locally compact Hausdorff and let $\mathcal{B} \subset C_0(X)$ be a subalgebra that satisfies the following properties.

(i) If $f \in \mathcal{B}$, then $\overline{f} \in \mathcal{B}$.

(ii) For all $x \in X$ there exists $f \in \mathcal{B}$ with $f(x) \neq 0$.

(iii) $\forall x \neq y, \exists f \in \mathcal{B}$ with $f(x) \neq f(y)$.

Then $\mathcal{B}$ is dense in $C_0(X)$ for $\|\cdot\|_\infty$.

We prove Theorem 4.7.

Proof of theorem 4.7. Since A is abelian, every $x \in A$ is normal and hence $\|\widehat{x}\|_\infty = \|x\|_{\mathrm{Sp}} = \|x\|$ by virtue of Theorem 2.9 and Proposition 4.2. Thus,

$$\begin{aligned} A &\longrightarrow C_0(\widehat{A}) \\ x &\longmapsto \quad \widehat{x} \end{aligned}$$

is norm-preserving. Now let $\mathcal{B} = \{\widehat{x} : x \in A\} \subset C_0(\widehat{A})$. Then $\mathcal{B}$ is a subalgebra, and since A is complete and $x \mapsto \widehat{x}$ is norm-preserving, $\mathcal{B}$ is complete and hence closed in $C_0(\widehat{A})$. Now we want to show that $\mathcal{B}$ satisfies the assumptions of the Stone-Weierstrass theorem. If that were the case, closedness and density would imply equality of $\mathcal{B}$ and $C_0(\widehat{A})$.

Step 1. First observe that if $a = a^*$, then $\widehat{a}(\widehat{A}) \subset \mathbb{R}$ since $\widehat{a}(\widehat{A}) \subset \mathrm{Sp}_A(a)$ by Theorem 3.23, and $\mathrm{Sp}_A(a) \subset \mathbb{R}$ by Corollary 4.6.

Step 2. Let $\chi \in \widehat{A}$. Since $\chi(A) = \mathbb{C}$, there exists $a \in A$ with $\chi(a) \neq 0$, that is $\widehat{a}(\chi) \neq 0$. Then $\mathcal{B}$ satisfies (ii).

Step 3. For every $\chi_1 \neq \chi_2$, by definition there exists $a \in A$ such that $\chi_1(a) \neq \chi_2(a)$, that is $\widehat{a}(\chi_1) \neq \widehat{a}(\chi_2)$. This proves that $\mathcal{B}$ satisfies (iii).

Finally, we conclude by the Stone-Weierstrass theorem that $\mathcal{B}$ is dense in $C_0(\widehat{A})$, and since it is closed, $\mathcal{B} = C_0(\widehat{A})$. □

Recall that we defined the radical J of A in the previous chapter by

$$J = \bigcap_{\chi \in \widehat{A}} \ker \chi.$$

Denote by $\Delta\colon A \to C_0(\widehat{A})$ the Guelfand transform. Its kernel is

$$\ker \Delta = \left\{x \in A : \widehat{x}(\chi) = 0,\ \forall \chi \in \widehat{A}\right\} = \bigcap_{\chi \in \widehat{A}} \{x \in A : \chi(x) = 0\} = \bigcap_{\chi \in \widehat{A}} \ker \chi = J.$$

Therefore, the radical of A is nothing else than the kernel of the Guelfand transform on A. But using the previous Theorem 4.7 we see that in the case of commutative C^*-algebras, the kernel $\ker \Delta$ is always trivial, so this implies that commutative C^*-algebras are always semisimple.

To conclude this section, we remark how it is natural to enquire about the functorial nature of the map $A \to C_0(\widehat{A})$. The following corollary of Theorem 4.7 answers this question.

Corollary 4.8. For two abelian C^*-algebras A and B, the following are equivalent.

(i) A and B are isomorphic as $\mathbb{C}$-algebras.

(ii) $\widehat{A}$ and $\widehat{B}$ are homeomorphic.

(iii) A and B are isomorphic as C^*-algebras.

Proof. Let $T\colon A \longrightarrow B$ be a $\mathbb{C}$-algebra isomorphism. Since B is semisimple, Corollary 3.18 implies that T is continuous. Since A is semisimple as well, T^{-1} is continuous by the same argument. From this it follows that if χ is a character in $\widehat{B}$, then $\chi \circ T \in \widehat{A}$, and the map

$$\begin{aligned} t\colon \widehat{B} &\longrightarrow \widehat{A} \\ \chi &\longmapsto \chi \circ T \end{aligned}$$

is a bijection. We are now going to show that it is continuous.

Let $\chi_0 \in \widehat{A}$, $a_1, \dots, a_n \in A$ and $\varepsilon > 0$. For $\chi \in \widehat{A}$,

$$|\chi(a_i) - \chi_0(a_i)| < \varepsilon \iff \left|\left(\chi \circ T^{-1}\right)(T(a_i)) - \left(\chi_0 \circ T^{-1}\right)(T(a_i))\right| < \varepsilon$$

for $1 \leq i \leq n$. This shows that

$$t^{-1}(\mathcal{U}(\chi_0; a_1, \ldots, a_n; \varepsilon)) = \mathcal{U}\big(t^{-1}(\chi_0); T(a_1), \ldots, T(a_n); \varepsilon\big) .$$

Therefore t is a homeomorphism. Thus the map

$$\begin{aligned} \delta\colon C_0(\widehat{A}) &\longrightarrow C_0(\widehat{B}) \\ f \quad &\longmapsto \; f \circ t \end{aligned}$$

is an isometric C^*-isomorphism, and $\delta \circ \wedge = \wedge \circ T$,

$$[\delta(\widehat{a})]\,(\chi) = (\widehat{a} \circ t)\,(\chi) = \widehat{a}(t(\chi)) = \widehat{a}(\chi \circ T) = (\chi \circ T)\,(a) = \chi(T(a)) = \widehat{T(a)}(\chi).$$

This shows that T is an isometric C^*-isomorphism. □

As a next application we will prove an "abstract spectral theorem" for normal operators. However, we first dedicate the rest of the chapter to deal with the following issue. If A is a Banach algebra with identity e and B is a sub-Banach algebra with $e \in B$, then clearly, if for $x \in B$ and $\lambda \in \mathbb{C}$, the element $x - \lambda e$ is not invertible in A, then it cannot be invertible in B. Thus

$$\mathrm{Sp}_A(x) \subset \mathrm{Sp}_B(x).$$

In general, equality does not hold. What we want to understand is if equality holds when we endow A and B with a bit more structure, i.e. when they are C^*-algebras. Before this, we illustrate the issue with an example.

Example 4.9. Let $A = \ell^1(\mathbb{Z})$, abelian Banach algebra with involution $f^*(x) := \overline{f(-x)}$. Let

$$B = \big\{f \in \ell^1(\mathbb{Z}) : f(x) = 0 \quad \forall x \leq -1\big\} \ni \delta_0.$$

Recall that δ_0 is an identity, and that $\operatorname{supp} f * g \subset \operatorname{supp}(f) + \operatorname{supp}(g)$ (as can be verified elementarily from the definition) resulting in B being a subalgebra of A. One can also check that $\delta_1 \in B \subset A$, with $\delta_1^* = \delta_{-1}$ and $\delta_1 * \delta_1^* = \delta_0$, so δ_1 is unitary. We leave it as an exercise to check that $\mathrm{Sp}_A(\delta_1) \subset \mathbb{T}$, for which we give the hint that one can use the fact that $\widehat{\ell^1(\mathbb{Z})} = \mathrm{Hom}(\mathbb{Z}, \mathbb{T})$ and that for $x \in A$, $\widehat{x}(\widehat{A}) = \mathrm{Sp}_A(x)$. Note however that δ_1 is not invertible in B, which means that $0 \in \mathrm{Sp}_B(x)$ and hence $\mathrm{Sp}_A(x)$ is a proper subset of $\mathrm{Sp}_B(x)$.

Proposition 4.10. Let $e \in B \subset A$ be unital C^*-algebras and let $x \in B$. Then $\mathrm{Sp}_A(x) = \mathrm{Sp}_B(x)$.

Proof. We have already observed that $\mathrm{Sp}_A(x) \subseteq \mathrm{Sp}_B(x)$. For the opposite inclusion, let $\lambda \notin \mathrm{Sp}_A(x)$. We will divide the proof into two steps.

Step 1. If x is self-adjoint, then by Proposition 4.5, $\mathrm{Sp}_B(x) \subset \mathbb{R}$. We may therefore assume that $\lambda \in \mathbb{R}$. Let $\varepsilon > 0$ and $\lambda_\varepsilon = \lambda + i\varepsilon$. Then λ_ε is not in $\mathrm{Sp}_B(x)$, and thus $(x - \lambda_\varepsilon e)^{-1} \in B$ for all $\varepsilon > 0$. By the continuity of the inverse in $G(A)$,

$$\lim_{\varepsilon \to 0}(x - \lambda_\varepsilon e)^{-1} = (x - \lambda e)^{-1} \in G(A).$$

On the other hand, since B is closed in A, we conclude that $(x - \lambda e)^{-1} \in B$. Hence, $\lambda \notin \mathrm{Sp}_B(x)$. This proves the result when x is self-adjoint.

Step 2. For the general case. Assume $y \in B$ is invertible in A. Then y^*y is invertible in A and hence in B as well, since it is self-adjoint. Therefore, there is $z \in B$ with $z(y^*y) = e$. The same argument applied to yy^* yields $u \in B$ with $(yy^*)u = e$. By setting $a = zy^*$ and $b = y^*u$ we find that $a, b \in B$ satisfy $ay = yb = e$. This immediately implies that $a = b = y^{-1}$, meaning that y is invertible in B. We conclude by applying this to $(x - \lambda e)$. □

Now we examine a particular result of intrinsic relevance. Let A be a unital C^*-algebra and let $x \in A$ be normal, that is $xx^* = x^*x$. Denote by $\mathbb{C}[X, Y]$ the polynomial algebra in two variables, and define

$$B = \overline{\{P(x, x^*) : P \in \mathbb{C}[X, Y]\}} \subset A,$$

with the convention that $x^0 = e$. Then B is an abelian sub C^*-algebra of A, and therefore $\text{Sp}_A(x) = \text{Sp}_B(x)$ by Proposition 4.10. For the statement of the theorem, we denote by $\mathbf{1}$ the constant function that sends every element to 1 in $\mathbb{C}$.

Theorem 4.11. The map

$$\begin{aligned} \widehat{x} \colon \widehat{B} &\longrightarrow \mathbb{C} \\ \chi &\longmapsto \chi(x) \end{aligned}$$

induces a homeomorphism $\widehat{B} \longrightarrow \text{Sp}_A(x)$. For every $f \in C\big(\text{Sp}_A(x)\big)$ there is a unique element in B which we denote by $f(x)$, satisfying

$$\widehat{f(x)}(\chi) = f(\chi(x)), \quad \forall \chi \in \widehat{B}.$$

The resulting map $C\big(\text{Sp}_A(x)\big) \longrightarrow B$ is a C^*-algebra isomorphism sending $\mathbf{1}$ to e and id to x.

Proof. We first note that the map $\widehat{x}$ is surjective by Theorem 3.23 and since the spectrum of A and the spectrum of B are equal by the previous Proposition 4.10. Moreover, $\widehat{x}$ is continuous since it is the Guelfand transform of $x \in A$. We now check that it is injective as well. For that, let $\chi_1, \chi_2 \in \widehat{B}$ with $\chi_1(x) = \chi_2(x)$. Then

$$\chi_1(x^*) = \overline{\chi_1(x)} = \overline{\chi_2(x)} = \chi_2(x^*),$$

and hence $\chi_1(P(x, x^*)) = \chi_2(P(x, x^*))$ for any $P \in \mathbb{C}[X, Y]$. By definition of B, these polynomials are dense in B, so this implies that $\chi_1 = \chi_2$, since χ_1, χ_2 are continuous. So $\chi \mapsto \chi(x)$ is a continuous bijection between compact Hausdorff spaces and hence it is a homeomorphism.

For $f \in C\big(\text{Sp}_A(x)\big)$ we have that the map $\chi \mapsto f \circ \widehat{x}(\chi) = f(\chi(x))$ is in $C(\widehat{B})$, and thus by the fact that the Guelfand transform on B is an isomorphism, as seen in Theorem 4.7, there is a unique element $b \in B$ with

$$\widehat{b}(\chi) = f(\chi(x)), \quad \forall \chi \in \widehat{B}.$$

If we denote b by $f(x)$, then the proof follows by some straightforward verifications. □

5 Spectral theorems

The material of this chapter is mostly taken from [Ru2] 12, and also from [EiWa] 12.5. Alternative sources are [EiWa] and [Zi].

As an application of the structure theorem for abelian C^*-algebras, namely Theorem 4.7, we are going to establish a spectral theorem for abelian sub-C^*-algebras of the C^*-algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. In particular, we obtain spectral theorems for normal operators. Observe that this result will be interesting with regards to the special case of the Fourier transform, since it is a unitary operator on L^2. We begin by presenting a brief reminder on measures and the Riesz representation theorem for positive functionals.

5.1 Measures

Throughout this section we will refer to X as a locally compact Hausdorff topological space, and we will denote the set of compactly supported, continuous functions on X by $C_{00}(X)$.

Definition 5.1. A positive linear functional on $C_{00}(X)$ is a $\mathbb{C}$-linear form $\Lambda\colon C_{00}(X) \longrightarrow \mathbb{C}$ such that whenever $f \in C_{00}(X)$ is such that $f(X) \subset [0, \infty)$, then $\Lambda(f) \geq 0$.

Now we recall the Riesz representation theorem for positive linear functionals.

Theorem 5.2 (*Riesz representation theorem*). Let X be locally compact, Hausdorff and $\Lambda\colon C_{00}(X) \longrightarrow \mathbb{C}$ be a positive linear functional. Then there is a σ-algebra $\mathcal{M}$ containing all Borel sets and a unique positive measure μ on $\mathcal{M}$ representing Λ in the following sense.

(i) $\Lambda(f) = \int_X f \,\mathrm{d}\mu, \quad f \in C_{00}(X)$.

(ii) $\mu(K) < \infty$ for all $K \subset X$ compact.

(iii) μ is outer regular, i.e. any set can be approximated in measure by outer open sets.

(iv) μ is inner regular on open sets and for sets of finite measure, i.e., these can be approximated in measure by inner compact sets.

(v) μ is complete, i.e. if $E \in \mathcal{M}$ with $A \subset E$ and $\mu(E) = 0$, then $A \in \mathcal{M}$.

Any measure satisfying the properties (ii) to (v) will be referred to as a positive ***regular***

Borel measure on X.

When X is compact then Λ is automatically continuous,

$$|\Lambda(f)| \leq \Lambda(f)\|f\|_\infty = \mu(X)\|f\|_\infty,$$

and therefore $\|\Lambda\| = \mu(X)$. This motivates the following definition.

Definition 5.3. Let X be compact and $\mathcal{B}$ the σ-algebra of Borel sets. A ***complex measure*** on $\mathcal{B}$ is a $\mathbb{C}$-linear combination of positive regular Borel measures.

We will only be interested in a few key features of these. An essential fact is the following: if μ is a complex measure and $E = \bigsqcup_{i\geq 1} E_i$ is a disjoint union of a countable collection of Borel sets, then

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i),$$

the series being absolutely convergent. For a more comprehensive treatment of complex measures, see chapter 8.

5.2 Operators in Hilbert spaces

In this section we will consider $\mathcal{H}$ to be a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$, and the norm given by $\|x\|^2 = \langle x, x\rangle$. Let us denote the distance between x and $c(t)$ by

$$d(x, c(t))^2 = \|x - c(t)\|^2,$$

where $c(t)$ is the straight line joining y and z, $c(t) = ty + (1-t)z$. A geometric fact is that when $y \neq z$, this function is strictly convex, and indeed its second derivative is $\|y - z\|^2$. Via this realisation we can prove the following result.

Theorem 5.4. Let $C \subset \mathcal{H}$ be a closed convex subset and $x \in \mathcal{H}$. Then there exists a ***unique*** point $y \in C$ with

$$d(x, y) = \inf \{d(x, z) : \; z \in C\}.$$

Recall that given $A \subset \mathcal{H}$, its orthogonal complement is defined as

$$A^\perp := \{x \in \mathcal{H} : \; \langle x, a\rangle = 0 \quad \forall a \in A\},$$

and it is a closed subspace of $\mathcal{H}$. Also, $\overline{A}^\perp = A^\perp$.

From this, together with Theorem 5.4, one can deduce the following result.

Theorem 5.5. Let $E \subset \mathcal{H}$ be a vector subspace. Then we have an orthogonal direct sum decomposition

$$\overline{E} \oplus E^\perp = \mathcal{H}.$$

Proof. For the proof, we have $E^\perp = \overline{E}^\perp$, so it suffices to prove the result in the case when E is closed. Fix some $x \in \mathcal{H}$ and pick $y \in E$ with

$$d(x, y) = \min \{d(x, z) : \; z \in E\},$$

then for every $m \in E$, the smooth function

$$d(t) := d(x, y + tm)^2$$

has a minimum at $t = 0$. Therefore, $0 = d'(0) = 2\,\mathrm{Re}\,\langle m, x - y\rangle$. Similarly, if we replace m with im we obtain $0 = \mathrm{Im}\,\langle m, x - y\rangle$. This shows $x - y \in E^\perp$. Hence we have the decomposition $x = y + (x - y)$ where $y \in E$ and $x - y \in E^\perp$. Since y is unique, it follows that the decomposition is unique and thus the sum $E + E^\perp = \mathcal{H}$ is direct. □

We have the following corollary.

Corollary 5.6. If $E \subset \mathcal{H}$ is a vector subspace, then $\left(E^\perp\right)^\perp = \overline{E}$.

Now we turn to some basic facts concerning bounded operators in $\mathcal{H}$, and in particular, normal operators.

Given $T \in \mathcal{L}(\mathcal{H})$, evaluation of $\langle Tx, x\rangle$ for every element $x \in \mathcal{H}$ determines $\langle Tx, y\rangle$ for $x, y \in \mathcal{H}$. This is given by the appropriate handling of the expressions

$$\begin{aligned}\langle T(x+y), x+y\rangle - \langle Tx, x\rangle - \langle Ty, y\rangle &= \langle Tx, y\rangle + \langle Ty, x\rangle \\ \langle T(x+iy), x+iy\rangle - \langle Tx, x\rangle - \langle Ty, y\rangle &= -i\,\langle Tx, y\rangle + i\,\langle Ty, x\rangle\,.\end{aligned}$$

The adjoint operator of T, denoted by T^*, and defined by

$$\langle T^* x, y\rangle = \langle x, Ty\rangle\,, \quad \forall x, y \in \mathcal{H}$$

induces an involution on the space $\mathcal{L}(\mathcal{H})$ via the application $T \mapsto T^*$ and in fact it endows this space with the structure of a C^*-algebra (see Example 1.6 in chapter 1).

Proposition 5.7. If $T \in \mathcal{L}(\mathcal{H})$, then

$$\ker T^* = (\mathrm{im}\,T)^\perp \quad \text{and} \quad \ker T = (\mathrm{im}\,T^*)^\perp.$$

Proof. The second assertion follows from the first since $(T^*)^* = T$. For the first one,

$$\begin{aligned} T^* y = 0 &\iff \langle x, T^* y\rangle = 0 \quad \forall x \in \mathcal{H} \\ &\iff \langle Tx, y\rangle = 0 \quad \forall x \in \mathcal{H} \\ &\iff y \in (\mathrm{im}\,T)^\perp. \end{aligned}$$

□

Let us now establish some basic facts about normal operators. Recall that $T \in \mathcal{L}(\mathcal{H})$ is normal if $TT^* = T^*T$.

Proposition 5.8. An operator $T \in \mathcal{L}(\mathcal{H})$ is normal if and only if $\|Tx\| = \|T^*x\|$ for every $x \in \mathcal{H}$. In addition, a normal operator has the following properties.

(i) $\ker T = \ker T^*$.

(ii) $\ker T = 0 \iff \overline{\mathrm{im}\,T} = \mathcal{H}$.

(iii) T is invertible if and only if there exists $c > 0$ such that

$$\|Tx\| \geq c\|x\|, \quad \forall x \in \mathcal{H}.$$

(iv) If $Tx = \alpha x$ for some $x \in \mathcal{H}$, then $T^*x = \bar{\alpha}x$.

(v) If $\alpha \neq \beta$ are eigenvalues of T, then the corresponding eigenspaces are orthogonal.

Example 5.9. In $\mathbb{C}^n$, a matrix X represents a normal operator if and only if there exists a unitary element $u \in U(n)$ such that uAu^{-1} is diagonal.

Proof. We begin by showing (i). For any $x \in \mathcal{H}$, we have

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle$$
$$\|T^*x\|^2 = \langle T^*x, T^*x \rangle = \langle TT^*x, x \rangle .$$

In combination with the remark following Corollary 5.6, this proves the first statement, from which $\ker T = \ker T^*$ readily follows.

For (ii), notice that by Proposition 5.7, $\ker T^* = (\operatorname{im} T)^\perp$, and by (i), we get that $\ker T = (\operatorname{im} T)^\perp$. Taking orthogonal spaces, we obtain $(\ker T)^\perp = \overline{\operatorname{im} T}$, and we conclude.

For (iii), we first prove the reverse implication. Assuming that $\|Tx\| \geq c\|x\|$ for all $x \in \mathcal{H}$ for some constant $c > 0$, it follows that $\ker T = 0$, and T is injective. This inequality gives us closedness of $\operatorname{im} T$, and in view of (ii) we find that $\operatorname{im} T = \mathcal{H}$, so T is also surjective. Therefore $T^{-1} \colon \mathcal{H} \longrightarrow \mathcal{H}$ exists as a linear map and $\|Tx\| \geq c\|x\|$ for every $x \in \mathcal{H}$, being equivalent to $\|y\| \geq c\|T^{-1}y\|$ for every $y \in \mathcal{H}$, provides continuity. The converse follows from the open mapping theorem.

For (iv), we merely check that $\ker(T - \alpha \operatorname{id})^* = \ker(T^* - \bar{\alpha} \operatorname{id})$, and by (i), this is $\ker(T - \alpha \operatorname{id})$.

Finally, for (v), set $Tx = \alpha x$, $Ty = \beta y$. Then a computation shows

$$\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle Tx, y \rangle = \langle x, T^*y \rangle = \left\langle x, \overline{\beta} y \right\rangle = \beta \langle x, y \rangle .$$

Therefore $(\alpha - \beta) \langle x, y \rangle = 0$, meaning that $\langle x, y \rangle = 0$ whenever $\alpha \neq \beta$. □

To conclude this section we discuss a few characterizations of self-adjoint projections. Recall that $P \in \mathcal{L}(\mathcal{H})$ is called a ***projection*** if $P^2 = P$.

Proposition 5.10. Let $P \in \mathcal{L}(\mathcal{H})$ be a projection. Then the following are equivalent:

(i) P is self-adjoint.

(ii) P is normal.

(iii) $\operatorname{im} P = (\ker P)^\perp$.

(iv) $\langle Px, x \rangle = \|Px\|^2 \quad \forall x \in \mathcal{H}$.

Moreover, for two self-adjoint projections P, Q we have

$$\operatorname{im} P \perp \operatorname{im} Q \iff PQ = 0.$$

Proof.

- (i) $\implies$ (ii): clear.
- (ii) $\implies$ (iii): $P^2 = P$ and P is normal. This implies that $\ker P = (\operatorname{im} P)^\perp$. Going to the orthogonal space, we get that $(\ker T)^\perp = (\operatorname{im} P)^{\perp\perp} = \overline{\operatorname{im} P}$. We claim that $\operatorname{im} P$ is closed: since P is a projection we have $x = Py$ for some $y \in \mathcal{H}$ if and only if $Px = x$. This means that $\operatorname{im} P = \ker(P - Id)$, so the former is closed.
- (iii) $\implies$ (iv): Assume now that $\operatorname{im} P = (\ker P)^\perp$ since $\ker P \oplus (\ker P)^\perp = \mathcal{H}$. This implies that $\operatorname{im} P \oplus \ker P = \mathcal{H}$. Now let $x = y + z$ with $y \in \operatorname{im} P$ and $z \in \ker P$, then

$$\langle Px, x\rangle = \langle Py + Pz, y + z\rangle = \langle y, y + z\rangle = \langle y, y\rangle = \langle Px, Px\rangle = \|Px\|^2.$$

- (iv) $\implies$ (i): Assume now that $\langle Px, x\rangle = \langle Px, Px\rangle \ \forall x \in \mathcal{H}$. Since the latter is real, by taking the complex conjugate we get $\langle x, P^*x\rangle = \langle P^*x, x\rangle \ \forall x \in \mathcal{H}$. We conclude by the remark following Corollary 5.6 that $P = P^*$.

The last claim is elementary. □

5.3 An example

Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator. We consider the abelian sub-C^*-algebra generated by T,

$$B := \overline{\{P(T, T^*) :\ P \in \mathbb{C}[X, Y]\}}.$$

Let $\operatorname{Sp}(T) \subset \mathbb{C}$ be the spectrum of T seen as an element of $\mathcal{L}(\mathcal{H})$. Then Theorem 4.11 provides us with a C^*-algebra isomorphism

$$\begin{aligned} C\big(\operatorname{Sp}(T)\big) &\longrightarrow B \\ f \quad &\longmapsto f(T) \end{aligned}$$

sending $\mathbf{1}$ to $\mathrm{id}_{\mathcal{H}}$ and id to T.

Lemma 5.11. If $\lambda_0 \in \operatorname{Sp}(T)$ is an isolated point, then λ_0 is an eigenvalue of T.

Proof. The characteristic function δ_{λ_0} of $\{\lambda_0\}$ is continuous on $\operatorname{Sp}(T)$. Using Theorem 4.11, let $P := \delta_{\lambda_0}(T)$. We have

$$\begin{aligned} \delta_{\lambda_0}\delta_{\lambda_0} &= \delta_{\lambda_0} \\ \delta_{\lambda_0} &= \overline{\delta_{\lambda_0}}, \end{aligned}$$

making P a self-adjoint projection. Now, observe that $(x - \lambda_0\mathbf{1}(x))\delta_{\lambda_0}(x) = 0$ for all $x \in \operatorname{Sp}(T)$, from where it follows that $(T - \lambda_0 \,\mathrm{id})P = 0$. Since $\delta_{\lambda_0} \neq 0$, we have $\operatorname{im} P \neq \{0\}$, and λ_0 is an eigenvalue of T. □

Example 5.12. Take now $\ell^2(\mathbb{Z})$ as the Hilbert space $\mathcal{H}$ and $Tf(x) := f(x+1)$ as the operator. Then $T^*f(x) = f(x-1)$ and T is unitary, so $\operatorname{Sp}(T) \subset \mathbb{T}$. We find that $\operatorname{Sp}(T)$ is compact, nonempty and without isolated points in $\mathbb{T}$. Therefore, it is in particular uncountable. Indeed, if it had an isolated point $\lambda_0 \in \operatorname{Sp}(T)$, then by Lemma 5.11 it would be an eigenvalue. Let $f \in \ell^2(\mathbb{Z})$ satisfy

$$f(x+1) = Tf(x) = \lambda_0 f(x) \quad \forall x \in \mathbb{Z}.$$

This would imply $f(x) = \lambda_0^x f(0)$ and

$$\sum_{x\in\mathbb{Z}} |f(x)|^2 = \sum_{x\in\mathbb{Z}} |\lambda_0|^x f(0),$$

and this is finite if and only if $f(0) = 0$, which implies $f \equiv 0$.

5.4 Resolutions of the identity

Let X be a locally compact Hausdorff space, $\mathcal{B}$ be the σ-algebra of Borel sets on X, and let $\mathcal{H}$ be a Hilbert space.

Definition 5.13. A ***resolution of the identity*** is a map $E\colon \mathcal{B} \longrightarrow \mathcal{L}(\mathcal{H})$ with the following properties.

(i) $E(\emptyset) = 0$, $E(X) = \mathrm{id}_{\mathcal{H}}$.

(ii) For every $\omega \in \mathcal{B}$, $E(\omega)$ is a self-adjoint projection.

(iii) $E(\omega_1 \cap \omega_2) = E(\omega_1)E(\omega_2)$.

(iv) If $\omega_1 \cap \omega_2 = \emptyset$, then $E(\omega_1 \cup \omega_2) = E(\omega') + E(\omega'')$.

(v) For all $x \in \mathcal{H}$ the set function $E_{x,x}(\omega) := \langle E(\omega)x, x\rangle$ is a positive regular Borel measure.

In a certain way, a resolution of identity is an object from which we can obtain a regular Borel measure on X. Part (v) in the definition shows how to do this for the particular case of a positive measure, but we will later see that it is possible extract more general measures. More generally, we can think of a resolution of identity itself as something similar to a measure that takes values in self-adjoint projections which will allow us to decompose certain operators into projections using an integral. In the finite dimensional case, the spectrum is discrete, so one can split up a matrix into projections onto its eigenspaces. In the infinite dimensional case the notion of spectrum entails more than only discrete parts and thus we need a more general tool to obtain a similar decomposition as in finite dimensions. This tool is the resolution of identity as we will see in the following sections.

Given two Borel sets ω_1 and ω_2 in X, disjointedness of ω_1 and ω_2 immediately implies by (iii) and Proposition 5.10 the orthogonality of the images of $E(\omega_1)$ and $E(\omega_2)$ as subspaces of $\mathcal{H}$. This is the reason behind requiring $E(\omega)$ to be a self-adjoint projection for any $\omega \in \mathcal{B}$. This, in particular, means that whenever ω_1 and ω_2 are disjoint, $E(\omega_1 \cup \omega_2)$ is also a projection, as one can readily see from the computations

$$(E(\omega_1) + E(\omega_2))^2 x = E(\omega_1)^2 x + E(\omega_1)E(\omega_2)x + E(\omega_2)E(\omega_1)x + E(\omega_2)^2 x = E(\omega_1)x + E(\omega_2)x$$

by the assumption $\omega_1 \cap \omega_2 = \emptyset$ and since $E(\omega)^2 = E(\omega \cap \omega) = E(\omega)$. In some sense, when ω_1 and ω_2 are disjoint, then $E(\omega_1 \cup \omega_2)$ is still a projection, using (iv) to see that the left-hand side is just $E(\omega_1 \cup \omega_2)^2$ and the right-hand side is $E(\omega_1 \cup \omega_2)$. We find that $\{E(\omega) : \omega \in \mathcal{B}\}$ is a family of commuting, self-adjoint projections.

Additionally, observe that

$$E_{x,x}(\omega) = \left\langle E(\omega)^2 x, x\right\rangle = \langle E(\omega)x, E(\omega)x\rangle = \|E(\omega)x\|^2 \geq 0.$$

The first 4 axioms imply that $E_{x,x}\colon \mathcal{B} \longrightarrow [0,\infty)$ is a positive additive set function taking values in $[0,1]$. What (v) essentially grants is the σ-additivity of this set function. Knowing this, we naturally wonder what happens when we take the scalar product with a pair of different elements $x, y \in \mathcal{H}$ in (v). Define $E_{x,y}(\omega) = \langle E(\omega)x, y\rangle$. By the remark following Corollary 5.6,

$$2E_{x,y}(\omega) = E_{x+y,x+y}(\omega) + iE_{x+iy,x+iy}(\omega) - (1+i)E_{x,x}(\omega) - (1+i)E_{y,y}(\omega),$$

which implies that $E_{x,y}$ is a complex measure, and enjoys σ-additivity (see chapter 8). Using this we show the following result.

Proposition 5.14. For every $x \in \mathcal{H}$, the map $\omega \mapsto E(\omega)x$ is countably additive, i.e. if $\omega = \bigsqcup_{n\geq 1} \omega_n$ is a countable, disjoint union of Borel sets, then

$$E(\omega)x = \sum_{n=1}^{\infty} E(\omega_n)x,$$

where the sum converges in the norm topology of $\mathcal{H}$.

To prove this, we will need the following lemma.

Lemma 5.15. Assume $\{x_n : n \geq 1\}$ is a sequence of pairwise orthogonal vectors in $\mathcal{H}$. Then the following are equivalent

(i) $\sum_{n=1}^{\infty} x_n$ converges in $\mathcal{H}$.

(ii) $\sum_{n=1}^{\infty} \|x_n\|^2$ converges.

(iii) $\sum_{n=1}^{\infty} \langle x_n, y\rangle$ converges for every $y \in \mathcal{H}$.

Proof. By orthogonality, for every $1 \leq n \leq m$,

$$\|x_1 + \cdots + x_m\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2$$

Assume (ii) holds. Then it suffices to check that

$$\left\|\sum_{n=N}^{N+m} x_n\right\|^2 = \sum_{n=N}^{N+m} \|x_n\|^2$$

to get that $\sum_{n=1} x_n$ is Cauchy if and only if $\sum_{n=1}\|x_n\|^2$ is, resulting in the equivalence of (i) and (ii).

To get (i) $\Longrightarrow$ (iii), we apply Cauchy-Schwarz,

$$\left|\sum_{k=n}^{m} \langle x_k, y\rangle\right| = \left|\left\langle \sum_{k=n}^{m} x_k, y\right\rangle\right| \leq \left\|\sum_{k=n}^{m} x_k\right\|\|y\|,$$

implying (iii).

Finally, for the implication (iii) $\Longrightarrow$ (ii), define the linear form Λ_n for $n \geq 1$ by

$$\Lambda_n(y) := \sum_{k=1}^{n} \langle y, x_k\rangle .$$

By the hypotheses, for every vector $y \in \mathcal{H}$, $\lim_{n\to\infty} \Lambda_n(y)$ exists. By Banach-Steinhaus, the sequence $(\|\Lambda_n\|)_{n\geq 1}$ is bounded. To conclude, observe that

$$\|\Lambda_n\| = \|\sum_{k=1}^{n} x_k\| = \left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2},$$

which gives the desired result. □

Now we turn to the proof of the proposition.

Proof of Proposition 5.14. We have seen that for every $x, y \in \mathcal{H}$, $E(\omega) = \langle E(\omega)x, y\rangle$ is a complex measure, and if $\omega = \bigsqcup_{n\geq 1} \omega_n$, then we have

$$E_{x,y}(\omega) = \langle E(\omega)x, y\rangle = \sum_{k=1}^{\infty} \langle E(\omega_k)x, y\rangle .$$

Observe that $\{E(\omega_k)x\}$ is a pairwise orthogonal family of vectors, therefore Lemma 5.15 applies and $\sum_{n=1}^{\infty} E(\omega_n)x$ converges in $\mathcal{H}$. We can then perform the otherwise highly illegal move

$$E_{x,y}(\omega) = \sum_{n=1}^{\infty} \langle E(\omega_n)x, y\rangle = \left\langle \sum_{n=1}^{\infty} E(\omega_n)x, y \right\rangle$$

Since the expression above holds for every $y \in \mathcal{H}$, we get $E(\omega)x = \sum_{n=1}^{\infty} E(\omega_n)x$. □

Let us observe that since for a self-adjoint projection P we always have that $\|P\|$ is either 0 or 1, the series $\sum_{n\geq 1} E(\omega_n)$ will not converge in $\mathcal{L}(\mathcal{H})$ unless all except for finitely many $E(\omega_n)$'s are 0.

Example 5.16. Consider $X = \{x_1, \ldots, x_n\}$ with $\mathcal{B} = \mathcal{P}(X)$, $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, and $P_i \colon \mathcal{H} \to \mathcal{H}$ the orthogonal projection onto $\mathcal{H}_i$, $i = 1, \ldots, n$. Define

$$E(\omega) := \sum_{x_i \in \omega} P_i.$$

Then E is a resolution of the identity.

5.5 The algebra $L^\infty(E)$

Throughout this section, consider X to be a compact, Hausdorff space, and let $E\colon \mathcal{B} \to \mathcal{L}(\mathcal{H})$ be a resolution of the identity. We proceed to define the C^*-algebra $L^\infty(E)$ of bounded Borel functions. Let $f\colon X \longrightarrow \mathbb{C}$ be a complex valued measurable function. We want to define the ***essential image*** of f.

Lemma 5.17. Let $\omega = \bigcup_{n\geq 1} \omega_n$ where $\omega_n \in \mathcal{B}$ with $E(\omega_n) = 0$ for all $n \geq 1$. Then $E(\omega) = 0$.

Proof. Since $E(\omega_k) = 0$, we get $E_{x,x}(\omega_n) = 0$ for every $n \geq 1$, $x \in \mathcal{H}$. By σ-additivity of $E_{x,x}$ we obtain $E_{x,x}(\omega) = 0$ for every $x \in \mathcal{H}$. Therefore, $E(\omega) = 0$. □

Let $\{D_n : n \in \mathbb{N}\}$ be a countable basis for the topology on $\mathbb{C}$, consisting of open discs. Define

$$V := \bigcup_{E(f^{-1}(D_n))=0} D_n.$$

The discs D_n with $E(f^{-1}(D_n)) = 0$ have a similar role as null sets in measure theory, in that their contributions with respect to the resolution of identity are negligible. Furthermore, V is clearly an open set of $\mathbb{C}$ satisfying additionally, by the preceding lemma, that $E(f^{-1}(V)) = 0$. It follows that V is the largest open subset of $\mathbb{C}$ with the property that $E(f^{-1}(V)) = 0$. We call the complement $\mathbb{C} \setminus V$ the ***essential image*** of f, denoted ess im$(f) \subset \mathbb{C}$. We say f is ***essentially bounded*** if ess im(f) is bounded, and we define the L^∞-norm of a function in terms of the essential image. Namely,

$$\|f\|_\infty := \sup \{|\lambda| : \lambda \in \text{ess im}(f)\}.$$

Exercise (*Support and essential range*)

Let $z \in$ ess im(f). Then for every open neighborhood D of z, we have $E(f^{-1}(D)) \neq 0$. This property is reminiscent of the notion of support of a measure.

Next, let $\mathcal{B}^\infty$ denote the space of bounded Borel functions, equipped with the norm $\|\cdot\|$,

$$\mathcal{B}^\infty := \{f \colon X \longrightarrow \mathbb{C} : \text{ Borel measurable, with } \|f\| := \sup\{|f(x)| : x \in X\} < \infty\}.$$

One verifies that $\mathcal{B}^\infty(X)$, endowed with $\|\cdot\|$, is in fact an abelian C^*-algebra for pointwise multiplication. The subspace

$$N = \{f \in \mathcal{B}^\infty(X) : \|f\|_\infty = 0\}.$$

is a closed ideal in $\mathcal{B}^\infty(X)$ and we define

$$L^\infty(E) := \mathcal{B}^\infty(X)/N.$$

Then the quotient norm of a class $[f] = f + N$ is just $\|f\|_\infty$, and the spectrum

$$\text{Sp}_{L^\infty(E)}([f]) = \text{ess im}(f).$$

It is customary to write f for an element $[f]$ of $L^\infty(E)$.

What we have done until this point can be viewed as a similar construction as the one of $L^\infty(\mathbb{R})$ with the Lebesgue measure. In this general setting, the set V is built by joining all the open sets that f builds up from sets of E-measure zero. This precisely states that the pushforward measure by f of V is zero. Also, V is taken by joining open sets which form a basis of the topology in $\mathbb{C}$ and this choice makes it the biggest possible set with this property. This is why we define $\mathbb{C} \setminus V$ to be the essential image of f: the rest of $\mathbb{C}$ gets positive measure from f. Then we simply collapse all functions with L^∞-norm equal to zero into one equivalence class, meaning that two functions are the same if their difference is zero over its essential image. This is the same as saying they differ only on a set of measure zero.

We can make an essential observation: for every $f \in \mathcal{B}^\infty(X)$, we have $\|f\|_\infty \leq \|f\|$. This is because ess im $f \subset$ im f.

Theorem 5.18. Given a resolution of the identity $E\colon \mathcal{B} \longrightarrow \mathcal{L}(\mathcal{H})$, there is a C^*-algebra isomorphism

$$\psi\colon L^\infty(E) \longrightarrow B \subset \mathcal{L}(\mathcal{H})$$

onto a sub-C^*-algebra B which is related to E by

$$\langle \psi(f)x, y\rangle = \int_X f \,\mathrm{d}E_{x,y},$$

where $x, y \in \mathcal{H}$ and $f \in L^\infty(E)$. Moreover,

$$\|\psi(f)x\|^2 = \int_X |f|^2 \,\mathrm{d}E_{x,x}$$

and an operator $Q \in \mathcal{L}(\mathcal{H})$ commutes with every $E(\omega)$ if and only if it commutes with B.

We will encode the equation $\langle \psi(f)x, y\rangle = \int_X f \,\mathrm{d}E_{x,y}$ by using the notation

$$\psi(f) = \int_X f \,\mathrm{d}E.$$

Notice that this result can be seen as a sort of Riesz representation theorem, where a functional is somewhat expressed as integration of the function it is evaluated on against a particular measure. Here, however, we have an inner product instead of just a functional.

Definition 5.19. A simple function on X is a function $s \in \mathcal{B}^\infty(X)$ taking finitely many values.

Let $\mathcal{S}(X)$ be the $\mathbb{C}$-vector space of simple functions. It is a sub-algebra of $\mathcal{B}^\infty(X)$, and it is dense for the norm topology. Now we move on to the proof of the theorem.

Proof of Theorem 5.18. Let $s \in \mathcal{S}(X)$ with distinct values $\alpha_1, \dots, \alpha_n$, so that

$$s = \sum_{i=1}^n \alpha_i \chi_{\omega_i},$$

with $\omega_i = s^{-1}(\alpha_i) \in B$. Define now

$$\Psi(s) = \sum_{i=1}^n \alpha_i E(\omega_i).$$

Then Ψ is a C^*-algebra map, that is,

(i) $\Psi(s)^* = \sum_{i=1}^n \overline{\alpha_i} E(\omega_i)^* = \sum_{i=1}^n \overline{\alpha_i} E(\omega_i) = \Psi(\sum_{i=1}^n \overline{\alpha_i} \chi_{\omega_i}) = \Psi(\bar{s})$.

(ii) $\Psi(st) = \Psi(s)\Psi(t)$ since for $s = \sum_{i=1}^n \alpha_i \chi_{\omega_i}$, $t = \sum_{j=1}^m \alpha_i \chi_{\omega'_j}$,

$$\Psi(st) = \Psi\left(\sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j E(\omega_i \cap \omega'_j)\right) = \Psi\left(\sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j E(\omega_i) E(\omega'_j)\right) = \Psi(s)\Psi(t)$$

Linearity is left as an exercise. Now, let us compute

$$\begin{aligned}\langle \Psi(s)x, y\rangle &= \left\langle \sum \alpha_i E(\omega_i)x, y \right\rangle = \sum \alpha_i \langle E(\omega_i)x, y\rangle = \sum \alpha_i \int_X \chi_{\omega_i} \, \mathrm{d}E_{x,y} \\ &= \int_X \left(\sum \alpha_i \chi_{\omega_i}\right) \mathrm{d}E_{x,y} = \int_X s \, \mathrm{d}E_{x,y}\end{aligned}$$

and we end up with

$$\|\Psi(s)x\|^2 = \langle \Psi(s)^*\Psi(s)x, x\rangle = \langle \Psi(\overline{s})\Psi(s)x, x\rangle = \left\langle \Psi(|s|^2)x, x \right\rangle = \int_X |s|^2 \, \mathrm{d}E_{x,x}.$$

This immediately implies

$$\|\Psi(s)x\|^2 \le \|s\|_\infty^2 \int_X \mathrm{d}E_{x,x} = \|s\|_\infty^2 \|x\|^2.$$

Moreover, if $x \in \mathrm{im}(E(\omega_j))$, then

$$\Psi(s)x = \alpha_j E(\omega_j)x = \alpha_j x$$

since the projections $E(\omega_i)$ have mutually orthogonal range. Therefore, if we chose j such that $|\alpha_j| = \|s\|_\infty$, then for $x \in \mathrm{im}\, E(\omega_j)$ we have $\|\Psi(s)x\|_\infty = \|s\|_\infty \|x\|$ and hence $\|\Psi\| = \|s\|_\infty$.

Let now $f \in \mathcal{B}^\infty(X)$ and $(s_n)_{n\ge1} \subset \mathcal{S}(X)$ with $\lim_{n\to\infty}\|s_n - f\| = 0$ (i.e. we approximate f by simple functions). Then it follows from the previous conclusion that since (s_n) is a Cauchy sequence, that $(\|\psi(s_n)\|)_{n\ge1}$ is a Cauchy sequence as well. Now let $\psi(f) = \lim_{n\to\infty} \psi(s_n)$. One then verifies easily that $\Psi(f)$ is independent of the approximating sequence of simple functions s_n, and $\|\Psi(s)\| = \|s\|_\infty \le \|s\|$ implies that $\|\Psi(f)\| = \|f\|_\infty$.

Now let $x \in \mathcal{H}$, and see that

$$\langle \Psi(f)x, x\rangle = \lim_{n\to\infty} \langle \Psi(s_n)x, x\rangle = \lim_{n\to\infty} \int_X s_n \, \mathrm{d}E_{x,x}.$$

Given that $\|s_n - f\|$ tends to zero, we can exchange the limit and the integral above to find that

$$\langle \Psi(f)x, x\rangle = \int_X f \, \mathrm{d}E_{x,x}.$$

An analogous argument shows that

$$\|\Psi(f)x\|^2 = \int_X |f|^2 \, \mathrm{d}E_{x,x}.$$

Hence Ψ is a C^*-injection of $L^\infty(E)$ into $\mathcal{L}(\mathcal{H})$, and its image A is therefore closed. The final assertion is then proved by an approximation argument. □

5.6 The spectral theorem

The spectral theorem establishes that every bounded, normal operator T over a Hilbert space $\mathcal{H}$ (i.e. $T \in \mathcal{L}(\mathcal{H})$) induces a resolution of the identity E in a canonical way. This is defined on the Borel subsets of the spectrum of T within $\mathbb{C}$, and in fact T can be reconstructed from E by an integration process as discussed in Theorem 5.18. In fact, this will be a special case of the spectral theorem for abelian sub-C^*-algebras of $\mathcal{L}(\mathcal{H})$.

Theorem 5.20. Let $A \subset \mathcal{L}(\mathcal{H})$ be an abelian sub-C^*-algebra containing $\mathrm{id}_{\mathcal{H}}$, and let $\widehat{A}$ be its Guelfand spectrum. Then the following hold.

(i) There is a unique resolution of the identity E defined on the Borel sets of $\widehat{A}$ which satisfies, for every $T \in A$,
$$T = \int_{\widehat{A}} \widehat{T}\, \mathrm{d}E,$$
where $\widehat{T} \in C(\widehat{A})$ is the Guelfand transform of T.

(ii) The inverse $C(\widehat{A}) \longrightarrow A$ of the Guelfand transform extends to a C^*-algebra isomorphism,
$$\Phi\colon L^\infty(E) \longrightarrow B$$
onto a sub-C^*-algebra $B \subset \mathcal{L}(\mathcal{H})$ with $B \supset A$ and given by
$$\Phi(f) = \int_{\widehat{A}} f\, \mathrm{d}E, \quad f \in L^\infty(E).$$
Explicitly, Φ is linear, multiplicative and satisfies both $\Phi(\bar{f}) = \Phi(f)^*$ and $\|\Phi(f)\| = \|f\|_\infty$.

(iii) B is the closure in $\mathcal{L}(\mathcal{H})$ of the space of all finite linear combinations of the projections $E(\omega)$.

(iv) If $\omega \subset \widehat{A}$ is open and non-empty, then $E(\omega) \neq 0$.

(v) An operator $S \in \mathcal{L}(\mathcal{H})$ commutes with every $T \in A$ if and only if it commutes with all the projections $E(\omega)$.

Before continuing onto the proof of the theorem, we will need one preliminary lemma, and we stop here to make a few remarks.

Remark.

(i) Let $f \in C(\widehat{A})$ and assume that $\|f\|_\infty = 0$. Then $E\Big(|f|^{-1}((0,\infty))\Big) = 0$ and $|f|^{-1}((0,\infty))$ being open, together with (iv), implies that it is empty, hence $f = 0$ and $C(\widehat{A})$ injects into $L^\infty(E)$.

(ii) Since A is abelian, it follows from (v) that every $T \in A$ commutes with every projection $E(\omega)$.

(iii) For $T \in A$ and $\chi_0 \in \widehat{A}$, let $\varepsilon > 0$ and consider
$$\omega = \left\{\chi \in \widehat{A} : \left|\widehat{T}(\chi) - \widehat{T}(\chi_0)\right| < \varepsilon\right\},$$
a non-empty open set. Then $E(\omega) \neq 0$ and we claim that $\forall v \in \operatorname{im} E(\omega)$, we have
$$\|T(v) - \widehat{T}(\chi_0)v\| \leq \varepsilon\|v\|,$$
so $\operatorname{im} E(\omega)$ consists of "almost eigenvectors" of the eigenvalue $\widehat{T}(\chi_0)$. Indeed,
$$(T - \widehat{T}(\chi_0)\,\mathrm{id})E(\omega) = \Phi((\widehat{T} - \widehat{T}(\chi_0)\mathbf{1})\chi_\omega),$$

and therefore

$$\left\|\left(T - \widehat{T}(\chi_0)\,\mathrm{id}\right) E(\omega)\right\| = \left\|\left(\widehat{T} - \widehat{T}(\chi_0)\mathbf{1}\right) \chi_\omega\right\|_\infty \leq \varepsilon$$

Now we introduce the lemma we will need to prove Theorem 5.20

Lemma 5.21. Let $f\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a sesquilinear, bounded form in the sense that

$$M := \sup\left\{|f(x,y)| : \|x\| = \|y\| = 1\right\} < \infty.$$

Then there is a unique $T \in \mathcal{L}(\mathcal{H})$ such that

$$f(x,y) = \langle Tx, y\rangle\,, \quad \forall x, y \in \mathcal{H}.$$

Moreover, $\|T\| = M$.

Proof. By scaling and sesquilinearity, we get that $|f(x,y)| \leq M\|x\|\|y\|$ for every $x, y \in \mathcal{H}$. Next, for every y there exists a unique $S(y) \in \mathcal{H}$ such that $f(x,y) = \langle x, S(y)\rangle$, by the Riesz representation theorem and since $x \mapsto f(x,y)$ for a given y is a continuous linear functional. Additionally, one easily checks that S is linear and bounded because f is sesquilinear. Defining $T = S^*$ yields the desired result, and notice that

$$\begin{aligned}\|S\| &= \sup_{\|y\|=1} \|Sy\| = \sup_{\|y\|=1} \sup_{\|x\|=1} \langle x, Sy\rangle \\ &= \sup_{\|y\|\leq 1} \sup_{\|x\|=1} |f(x,y)| = M.\end{aligned}$$ ☐

We finally move onto the proof of the theorem.

Proof of Theorem 5.20. Let us denote $g\colon C(\widehat{A}) \longrightarrow A$ the inverse of the Guelfand isomorphism. Then naturally g is also a C^*-algebra isomorphism. Pick $x \in \mathcal{H}$ and consider

$$\begin{aligned} C(\widehat{A}) &\longrightarrow \mathbb{C} \\ f &\longmapsto \langle g(f)x, x\rangle\,. \end{aligned}$$

This is a linear functional on $C(\widehat{A})$. Moreover, assume $f \geq 0$ and set $h := \sqrt{f}$. Then $f = \bar{h}h$, and we get $g(f) = g(h)^* g(h)$, from which it follows that

$$\langle g(f)x, x\rangle = \langle g(h)^* g(h)x, x\rangle = \|g(h)x\|^2 \geq 0.$$

With this, the Riesz representation theorem immediately gives a Borel measure $E_{x,x}$ on $\widehat{A}$ satisfying

$$\langle Tx, x\rangle = \int_{\widehat{A}} \widehat{T}\,\mathrm{d}E_{x,x}, \quad \forall T \in A,\ \forall x \in \mathcal{H}.$$

We can now define a complex measure, which will be useful since in the end we need a resolution of the identity. Define by polarization

$$E_{x,y} := \frac{1}{2}(E_{x+y,x+y} + iE_{x+iy,x+iy} - (1+i)E_{x,x} - (1+i)E_{y,y}).$$

By the polarization property of the inner product we directly get

$$\langle Tx, y\rangle = \int_{\widehat{A}} \widehat{T}\,\mathrm{d}E_{x,y}, \quad \forall T \in A,\ \forall x, y \in \mathcal{H}.$$

This last equality implies that for every $f \in C(\widehat{A})$ the function

$$(x, y) \mapsto \int_{\widehat{A}} \widehat{T}\,\mathrm{d}E_{x,y}$$

is sesquilinear. Since complex measures are completely determined by their value on $C(\widehat{A})$, it follows that this holds for every $f \in \mathcal{B}^\infty(\widehat{A})$. Next, observe that for every $v \in \mathcal{H}$ and $f \in \mathcal{B}^\infty(\widehat{A})$, we have

$$\left|\int_{\widehat{A}} f\,\mathrm{d}E_{v,v}\right| \le \|f\| E_{v,v}(\widehat{A}) = \|f\|\|v\|^2.$$

Using the definition of $E_{x,y}$ we get that the sesquilinear form $(x, y) \mapsto \int_{\widehat{A}} f\,\mathrm{d}E_{x,y}$ is bounded over functions in $\mathcal{B}^\infty(\widehat{A})$. By Lemma 5.21, there exists $\Phi(f) \in \mathcal{L}(\mathcal{H})$ with

$$\langle \Phi(f)x, y\rangle = \int_{\widehat{A}} f\,\mathrm{d}E_{x,y}, \quad \forall f \in \mathcal{B}^\infty(\widehat{A}).$$

We now need to analyse the properties of Φ in order to conclude the proof. We can sum these up as follows.

(i) Φ is clearly ***linear***.

(ii) From $\langle Tx, y\rangle = \int_{\widehat{A}} \widehat{T}\,\mathrm{d}E_{x,y} = \left\langle \Phi(\widehat{T})x, y\right\rangle$ we deduce that $\Phi(\widehat{T}) = T$ for every $T \in A$, making it clear that Φ ***extends*** the inverse of the Guelfand transform to the whole $\widehat{A}$.

(iii) A few computations show that $\Phi(f)$ is a C^****-homomorphism***,

$$\langle \Phi(f)^*x, x\rangle = \langle x, \Phi(f)x\rangle = \overline{\langle \Phi(f)x, x\rangle} = \overline{\int_{\widehat{A}} f\,\mathrm{d}E_{x,x}} = \int_{\widehat{A}} \overline{f} E_{x,x} = \left\langle \Phi(\overline{f})x, x\right\rangle.$$

Since $x \in \mathcal{H}$ was arbitrary, we deduce that $\Phi(f)^* = \Phi(\bar{f})$ for every $f \in \mathcal{B}^\infty(\widehat{A})$.

(iv) To show ***multiplicativity*** of Φ, we use the fact that the Guelfand transform is multiplicative. Let $S, T \in A$, and notice that $\widehat{TS} = \widehat{T}\widehat{S}$. Therefore

$$\int_{\widehat{A}} \widehat{S}\widehat{T}\,\mathrm{d}E_{x,y} = \int_{\widehat{A}} \widehat{ST}\,\mathrm{d}E_{x,y} = \langle S(T(x)), y\rangle = \int_{\widehat{A}} \widehat{S}\,\mathrm{d}E_{T(x),y}$$

Now, what this means is that the complex measures $\widehat{T}\,\mathrm{d}E_{x,y}$ and $\mathrm{d}E_{T(x),y}$ coincide on continuous functions over $\widehat{A}$, so they coincide on $\mathcal{B}^\infty(\widehat{A})$ as well, making it possible to replace $\widehat{S}$ by any $f \in \mathcal{B}^\infty(\widehat{A})$. Rewriting this yields

$$\begin{aligned}\int_{\widehat{A}} f\widehat{T}\,\mathrm{d}E_{x,y} &= \int_{\widehat{A}} f\,\mathrm{d}E_{T(x),y} = \langle \Phi(f)T(x), y\rangle = \langle T(x), \Phi(f)^*y\rangle \\ &= \langle T(x), z\rangle = \int_{\widehat{A}} \widehat{T}\,\mathrm{d}E_{x,z}\end{aligned}$$

where we have set $z = \Phi(f)^*y$. Therefore the complex measures $f\,\mathrm{d}E_{x,y}$ and $\mathrm{d}E_{x,z}$ coincide on $C(\widehat{A})$, hence on $\mathcal{B}^\infty(\widehat{A})$. That is,

$$\int_{\widehat{A}} fg\,\mathrm{d}E_{x,y} = \int_{\widehat{A}} g\,\mathrm{d}E_{x,z}.$$

This implies that

$$\begin{aligned}\langle \Phi(fg)x, y\rangle &= \int_{\widehat{A}} fg\,\mathrm{d}E_{x,y} = \int_{\widehat{A}} g\,\mathrm{d}E_{x,z} = \langle \Phi(g)x, z\rangle \\ &= \langle \Phi(g)x, \Phi(f)^*y\rangle = \langle \Phi(f)\Phi(g)x, y\rangle\,,\end{aligned}$$

which finally shows that

$$\Phi(fg) = \Phi(f)\Phi(g), \quad f, g \in \mathcal{B}^\infty(\widehat{A}).$$

Now we can define E in the following way. Let $\omega \subset \widehat{A}$ be a Borel subset, and set

$$E(\omega) = \Phi(\chi_\omega)$$

where, as usual, χ_ω represents the indicator function of ω. It remains to verify that E is a resolution of the identity.

(i) We readily see that $E(\emptyset) = \Phi(0) = 0$ and $E(\widehat{A}) = \Phi(\mathbf{1}) = id_{\mathcal{H}}$ since

$$\langle \Phi(\mathbf{1})x, x\rangle = \int_{\widehat{A}} \mathrm{d}E_{x,x} = \|x\|^2 = \langle x, x\rangle\,.$$

(ii) If ω is a Borel set, then $\chi_\omega^2 = \chi_\omega$ and $\chi_\omega = \overline{\chi_\omega}$, and since Φ is a C^*-homomorphism, we deduce that $E(\omega)$ is a self-adjoint projection. Indeed,

$$E(\omega)^2 = \Phi(\chi_\omega)^2 = \Phi(\chi_\omega^2) = \Phi(\chi_\omega) = E(\omega),$$

and

$$E(\omega)^* = \Phi(\chi_\omega)^* = \Phi(\overline{\chi_\omega}) = \Phi(\chi_\omega) = E(\omega).$$

(iii) For ω_1, ω_2 Borel sets,

$$E(\omega_1 \cap \omega_2) = \Phi(\chi_{\omega_1} \cdot \chi_{\omega_2}) = \Phi(\chi_{\omega_1})\Phi(\chi_{\omega_2}) = E(\omega_1) \cdot E(\omega_2).$$

(iv) If $\omega_1 \cap \omega_2 = \emptyset$, then $\chi_{\omega_1 \cup \omega_2} = \chi_{\omega_1} + \chi_{\omega_2}$ and we easily obtain $E(\omega_1 \cup \omega_2) = E(\omega_1) + E(\omega_2)$ by exploiting the additivity of Φ.

(v) $E_{x,x}(\omega) = \langle E(\omega)x, x\rangle$ so $E_{x,x}$ is a regular Borel measure by construction.

Therefore, E is a resolution of identity and

$$\langle \Phi(f)x, y\rangle = \int_{\widehat{A}} f\,\mathrm{d}E_{x,y}, \quad \forall f \in \mathcal{B}^\infty(\widehat{A}).$$

It follows from Theorem 5.18 that Φ factors via the projection $\mathcal{B}^\infty(\widehat{A}) \longrightarrow L^\infty(E)$ and induces the map $\psi\colon L^\infty(E) \longrightarrow \mathcal{L}(\mathcal{H})$ shown in Theorem 5.18. Thus, $\|\Phi(f)\| = \|f\|_\infty$, which shows assertions (i) and (ii) of the theorem.

We are left to prove (iii), (iv) and (v). Now, (iii) easily follows since every $L^\infty(E)$ function is a uniform limit of simple functions. For (iv), assume that $\omega \subset \widehat{A}$ is open and has $E(\omega) = 0$. Then $E_{x,x}|_\omega$ is the zero measure for every $x \in \mathcal{H}$, and given $T \in A$ with $\operatorname{supp} \widehat{T} \subset \omega$, this means that

$$\langle Tx, Tx \rangle = \int_\omega |\widehat{T}|^2 \, \mathrm{d}E_{x,x} = 0, \quad \forall x \in \mathcal{H}.$$

Hence $T = 0$, but this implies $\omega = \emptyset$.

Finally, to prove (v), let $S \in \mathcal{L}(\mathcal{H})$ and $T \in A$. Then

$$\langle STx, y \rangle = \langle Tx, S^*y \rangle = \int \widehat{T} \, \mathrm{d}E_{x,S^*y}$$

and

$$\langle TSx, y \rangle = \int \widehat{T} \, \mathrm{d}E_{Sx,y}.$$

Then

$$\langle STx, y \rangle = \langle TSx, y \rangle, \quad \forall x, y \in \mathcal{H} \Longleftrightarrow \mathrm{d}E_{x,S^*y} = \mathrm{d}E_{Sx,y} \quad \forall x, y \in \mathcal{H}.$$

This can be easily shown to be equivalent to

$$\langle E(\omega)x, S^*y \rangle = \langle E(\omega)Sx, y \rangle,$$

and since the right hand side is equal to $\langle SE(\omega)x, y \rangle$, we have finished and the proof is complete. □

Now we turn to a consequence of the spectral theorem which says that any normal operator is, up to a Hilbert space isomorphism, given by multiplication on an L^2-space.

Theorem 5.22. Let $\mathcal{H}$ be a separable Hilbert space and $A \subset \mathcal{L}(\mathcal{H})$ an abelian sub-C^*-algebra containing $\mathrm{id}_\mathcal{H}$. Then there is a finite positive regular Borel measure μ on $\widehat{A} \times \mathbb{N}$ and a Hilbert space isomorphism

$$\Lambda \colon \mathcal{H} \longrightarrow L^2(\widehat{A} \times \mathbb{N}, \mu)$$

such that for all $T \in A$ and $v \in \mathcal{H}$,

$$\Lambda(Tv)\,(\chi, n) = \widehat{T}(\chi)\Lambda(v)(\chi, n).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Lambda} & L^2(\widehat{A} \times \mathbb{N}, \mu) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle M_{\widehat{T}}} \\ \mathcal{H} & \xrightarrow{\Lambda} & L^2(\widehat{A} \times \mathbb{N}, \mu) \end{array}$$

where $M_{\widehat{T}} \colon L^2\left(\widehat{A} \times \mathbb{N}, \mu\right) \longrightarrow L^2\left(\widehat{A} \times \mathbb{N}, \mu\right)$ denotes the multiplication operator defined for $f \in L^2\left(\widehat{A} \times \mathbb{N}, \mu\right)$ by

$$(M_{\widehat{T}} f)(\chi, n) = \widehat{T}(\chi) f(\chi, n).$$

If $A \subset \mathcal{L}(\mathcal{H})$ is any sub-C^*-algebra (not necessarily abelian) then for every $v \in \mathcal{H}$, the space

$$A \cdot v = \{T(v) : T \in A\}$$

is a $\mathbb{C}$-vector subspace of $\mathcal{H}$, and so is its closure $\overline{A \cdot v}$. With this remark at hand, we introduce the following lemma concerning the decomposition of $\mathcal{H}$ into several closed, orthogonal subspaces given by A, which we will need to prove Theorem 5.22.

Lemma 5.23. Let $\mathcal{H}$ be a separable Hilbert space and $A \subset \mathcal{L}(\mathcal{H})$ as above. Then there exists a finite or countable family of vectors $v_1, v_2, v_3, \ldots$ such that the closed subspaces $\overline{Av_i}$ are pairwise orthogonal and

$$\mathcal{H} = \widehat{\bigoplus}_{i \geq 1} \overline{Av_i},$$

where $\widehat{\bigoplus}$ denotes the direct orthogonal sum.

Sketch of the proof. Consider the set

$$\mathcal{C} = \left\{F \subset \mathcal{H} \setminus \{0\} : \forall v \neq w \in F, \overline{Av} \perp \overline{Aw}\right\},$$

equipped with the partial order given by inclusion. Notice first that $\mathcal{C} \neq \emptyset$. Also, by Zorn's lemma, there exists a maximal set $F \in \mathcal{C}$. Now we claim that the direct sum $\bigoplus_{v \in F} \overline{A \cdot v}$ is dense in $\mathcal{H}$. This implies that $\widehat{\bigoplus}_{v \in F} \overline{A \cdot v} = \mathcal{H}$. Indeed, if it were not, define $\mathcal{L} = \bigoplus_{v \in F} \overline{A \cdot u}$. Then $\mathcal{L}^\perp \neq (0)$. By taking any vector $u \in \mathcal{L}$ and appending it to F we get a contradiction to its maximality. Indeed for any $v \neq u \in F$, $T \in A$,

$$\langle Tu, v\rangle = \langle u, T^* v\rangle = 0$$

since $T^* \in A$ as well. Finally since $\mathcal{H}$ is separable, F is countable. □

Proof of Theorem 5.22. Let E be the resolution of identity on $\widehat{A}$ given by Theorem 5.18. In particular, if $\mathcal{H} \ni v \neq 0$, then

$$\langle Tv, v\rangle = \int_{\widehat{A}} \widehat{T} \, \mathrm{d}E_{v,v} \quad \forall T \in A.$$

Applying this to $S = T^*T$,

$$\|Tv\|^2 = \langle T^*Tv, v\rangle = \int_{\widehat{A}} \left|\widehat{T}\right|^2 \mathrm{d}E_{v,v}.$$

What this relation implies is that $Tv = 0$ if and only if $\widehat{T} = 0$ almost everywhere with respect to $E_{v,v}$. Hence we obtain a well defined map

$$\begin{aligned} \mathcal{L}_v \colon A \cdot v &\longrightarrow L^2(\widehat{A}, E_{v,v}) \\ T \cdot v &\longmapsto \quad \widehat{T}. \end{aligned}$$

which is isometric ($\mathcal{L}_v$ is linear and norm preserving), and thus extends to a Hilbert space isomorphism between the respective closures since $C(\widehat{A})$ is dense in $L^2(\widehat{A}, E_{v,v})$, which means that $\mathcal{L}_v$ extends to $\overline{A \cdot v}$,

$$\mathcal{L}_v \colon \overline{A \cdot v} \longrightarrow L^2(\widehat{A}, E_{v,v}).$$

In addition, for every $a, T \in A$, we have

$$\mathcal{L}_v(aT \cdot v) = \widehat{(a \cdot T)} = \widehat{a} \cdot \widehat{T} = \widehat{a}\mathcal{L}_v(T \cdot v).$$

Therefore this extends to the closure, and for all $w \in \overline{A \cdot u}$,

$$\mathcal{L}_v(aw) = \widehat{a}\mathcal{L}_v(w).$$

Let now $v_1, v_2, \dots$ be a sequence of non-zero vectors such that

$$\mathcal{H} = \widehat{\bigoplus}_{i \geq 1} \overline{A \cdot v_i}$$

is an orthogonal decomposition. Let us scale the v_i's such that $\sum_{i \geq 1} \|v_i\| < \infty$. On $\widehat{A} \times \mathbb{N}$, we can define

$$\mu = \sum_{n \geq 1} E_{v_n, v_n} \otimes \delta_n,$$

and it becomes clear that if $f \in C(\widehat{A} \times \mathbb{N})$, then

$$\int f \, \mathrm{d}\mu = \sum_{n \geq 1} \int_{\widehat{A}} f(\chi, n) \, \mathrm{d}E_{v_n, v_n}(\chi).$$

The measure μ is a positive, regular Borel measure on $\widehat{A} \times \mathbb{N}$, and it is finite by the choice of the v_n, which follows from an easy computation. Define

$$\Lambda = \bigoplus_{n \geq 1} \mathcal{L}_{v_n} : \widehat{\bigoplus}_{n \geq 1} \overline{A \cdot v_n} \longrightarrow L^2\left(\widehat{A} \times \mathbb{N}, \mu\right).$$

In other words, for $v = \sum_n w_n$, where $w_n \in \overline{A \cdot v_n}$,

$$\Lambda(v)\,(\chi, m) = \sum_{m \geq 1} \mathcal{L}_{v_m}(w_m)(\chi).$$

We finally compute

$$\left\| \Lambda\left(\sum w_n \right) \right\|^2 = \sum_{m \geq 1} \int |\mathcal{L}_{v_m}(w_m)(\chi)|^2 \, \mathrm{d}E_{v_m, v_m}(\chi) = \sum \|w_m\|^2.$$

This shows that Λ is an isomorphism and it is an easy exercise to check that it satisfies the rest of the conclusions, namely that $\Lambda(Tv)(\chi, u) = \widehat{T}(\chi)\Lambda(v)(\chi, u)$. □

If we want to apply the spectral theorem to a single normal operator $T \in \mathcal{L}(\mathcal{H})$, we will use Theorem 4.11 and the construction preceding it. Namely, we consider

$$B := \overline{\{P(T, T^*) : \; P \in \mathbb{C}[X, Y]\}},$$

the abelian sub-C^*-algebra generated by id, T and T^*, and recall that here $\mathrm{Sp}(T)$ refers to $\mathrm{Sp}_{\mathcal{L}(\mathcal{H})}(T) = \mathrm{Sp}_B(T)$ (the latter equality following from Proposition 4.10). The evaluation map given by Theorem 4.11

$$\begin{aligned} \mathrm{ev} \colon \widehat{B} &\longrightarrow \mathrm{Sp}(T) \\ \chi &\longmapsto \chi(T) \end{aligned}$$

is then a homeomorphism. Furthermore, we have shown that, given $f \in C(\mathrm{Sp}(T))$, $f(T)$ denotes the unique element in B such that

$$\widehat{f(T)}(\chi) = f(\chi(T))\,.$$

The resulting map

$$\begin{aligned} C\big(\mathrm{Sp}(T)\big) &\longrightarrow B \\ f &\longmapsto f(T) \end{aligned}$$

is a C^*-algebra isomorphism. Observe that the identity above can also be written as

$$\widehat{f(T)}\big(\mathrm{ev}^{-1}(\lambda)\big) = f(\lambda) \quad \forall \lambda \in \mathrm{Sp}(T).$$

The spectral theorem then provides a unique resolution of the identity satisfying

$$b = \int_{\widehat{B}} \widehat{b}\, \mathrm{d}E, \quad \forall b \in B.$$

Then, for a Borel set $\omega \subset \mathrm{Sp}(T)$, define

$$E'(\omega) := E(\mathrm{ev}^{-1}(\omega)).$$

It is clear that E' is a resolution of the identity on $\mathrm{Sp}(T)$ (this would also work for any continuous function). Observe that since $\mathrm{ev}\colon \widehat{B} \longrightarrow \mathrm{Sp}(T)$ is continuous, we can define, for every positive regular Borel measure μ on $\widehat{B}$, its pushforward measure

$$\mathrm{ev}_* \,\mu(\omega) = \mu\big(\mathrm{ev}^{-1}(\omega)\big)\,,$$

which produces a regular, positive Borel measure $\mathrm{ev}_* \,\mu$ on $\mathrm{Sp}(T)$. Using that ev is a homeomorphism, one checks that, given any $f \in C(\widehat{B})$,

$$\int_{\widehat{B}} f(\chi)\, \mathrm{d}\mu(\chi) = \int_{\mathrm{Sp}(T)} (f \circ \mathrm{ev}^{-1})(y)\, \mathrm{d}(\mathrm{ev}_* \,\mu)(y).$$

With these remarks at hand, one deduces from $E'(\omega) = E\big(\mathrm{ev}^{-1}(\omega)\big)$ that

$$E'_{x,x}(\omega) = \big\langle E'(\omega)x, x\big\rangle = \big\langle E\big((\mathrm{ev})^{-1}(\omega)\big)\, x, x\big\rangle = \mathrm{ev}_*(E_{x,x})\,(\omega)$$

for every $x \in \mathcal{H}$. Now, for every $b \in B$,

$$\langle bx, x\rangle = \int_{\widehat{B}} \widehat{b}(\chi)\, \mathrm{d}E_{x,x}(\chi) = \int_{\mathrm{Sp}(T)} \left(\widehat{b} \circ (\mathrm{ev})^{-1}\right)(y)\, \mathrm{d}(\mathrm{ev}_* \,E_{x,x})\,(y).$$

Apply the above to $b = f(T)$ to obtain

$$\langle f(T)x, x\rangle = \int_{\mathrm{Sp}(T)} f(\lambda)\, \mathrm{d}E'_{x,x}(\lambda).$$

Thus we conclude from the spectral theorem that there is a unique resolution of the identity E' on $\mathrm{Sp}(T)$ such that

$$f(T) = \int_{\mathrm{Sp}(T)} f(\lambda)\, \mathrm{d}E'(\lambda) \quad \forall f \in C(\mathrm{Sp}(T)).$$

It is important to realize that $f(T)$ takes a concrete form when $f(\lambda) = p(\lambda, \overline{\lambda})$, where $p \in \mathbb{C}[X, Y]$ is a polynomial. The reason is the following: Theorem 4.11 asserts that $f \to f(T)$ is a $C*$-isomorphism from $C(\mathrm{Sp}(T))$ to B sending $\mathbf{1} \mapsto \mathrm{id}_{\mathcal{H}}$, and $\mathrm{id} \mapsto T$, and hence $\overline{\mathrm{id}} \mapsto T^*$. Therefore it sends $p(\lambda, \overline{\lambda}) \mapsto p(T, T^*)$, and

$$p(T, T^*) = \int_{\mathrm{Sp}(T)} p(\lambda, \overline{\lambda})\, \mathrm{d}E'(\lambda).$$

and this implies that E' is unique. Indeed, if E'' were another resolution of the identity on $\mathrm{Sp}(T)$ such that

$$T = \int_{\mathrm{Sp}(T)} \lambda\, \mathrm{d}E''(\lambda),$$

then it follows that the C^*-algebra map $\Psi : L^\infty(E'') \longrightarrow \mathcal{L}(\mathcal{H})$ from Theorem 5.18 sends id to T, hence $\overline{\mathrm{id}}$ gets sent to T^*, and as a result,

$$p(T, T^*) = \int_{\mathrm{Sp}(T)} p(\lambda, \overline{\lambda})\, \mathrm{d}E''(\lambda).$$

Since the functions of the form $\lambda \mapsto p(\lambda, \overline{\lambda})$ are dense in $C\big(\mathrm{Sp}(T)\big)$, we get that

$$f(T) = \int_{\mathrm{Sp}(T)} f(\lambda)\, \mathrm{d}E''(\lambda) \quad \forall f \in C\big(\mathrm{Sp}(T)\big)\,,$$

which readily implies $E'' = E'$. We have then essentially proved the following result.

Corollary 5.24. Let $T \in \mathcal{L}(\mathcal{H})$ be normal and $\mathrm{Sp}(T) \subset \mathbb{C}$ be its spectrum. There is a unique resolution of identity E on $\mathrm{Sp}(T)$ such that

$$T = \int_{\mathrm{Sp}(T)} \lambda\, \mathrm{d}E(\lambda).$$

Moreover, if $S \in \mathcal{L}(\mathcal{H})$ commutes with T and T^*, it commutes with every projection $E(\omega)$, for every Borel set $\omega \subset \mathrm{Sp}(T)$.

Remark. In the statement of the corollary we need S to commute with both T and T^* at the same time. However, one can show that if $T \in \mathcal{L}(\mathcal{H})$ is normal and S commutes with T, then it commutes with T^*. See [Ru2] 12.6.

Next, if E is the resolution of identity on $\mathrm{Sp}(T)$, we have seen that for any $f \in C(\mathrm{Sp}(T))$,

$$f(T) = \int_{\mathrm{Sp}(T)} f(\lambda)\, \mathrm{d}E(\lambda).$$

The map $\mathcal{B}^\infty(\mathrm{Sp}(T)) \longrightarrow L^\infty(E) \longrightarrow \mathcal{L}(\mathcal{H})$ given by Theorem 5.18 is related to E by the identity

$$\Psi(f) = \int_{\mathrm{Sp}(T)} f(\lambda)\, \mathrm{d}E(\lambda).$$

Hence it extends

$$\begin{aligned} C\big(\mathrm{Sp}(T)\big) &\longrightarrow B \\ f &\longmapsto f(T) \end{aligned}$$

and we will thus, for $f \in \mathcal{B}^\infty(\mathrm{Sp}(T))$, denote $\Psi(f)$ by $f(T)$. Therefore, for normal operators we have extended the continuous functional calculus to the bounded Borel functional calculus, and we have the following result.

Corollary 5.25. The map

$$\begin{aligned} \mathcal{B}^\infty\big(\mathrm{Sp}(T)\big) &\longrightarrow \mathcal{L}(\mathcal{H}) \\ f &\longmapsto f(T) \end{aligned}$$

is a C^*-homomorphism sending $\mathbf{1}$ to $\mathrm{id}_{\mathcal{H}}$ and id to T, and

$$\|f(T)\| \leq \|f\| = \sup\left\{|f(\lambda)| : \lambda \in \mathrm{Sp}(T)\right\}.$$

If $f \in C(\mathrm{Sp}(T))$, then $\|f(T)\| = \|f\|$. Moreover,

$$\|f(T)x\|^2 = \int_{\mathrm{Sp}(T)} |f|^2 \, \mathrm{d}E_{x,x},$$

and T is the limit in the norm topology of finite linear combinations of projections $E(\omega)$.

5.7 Schur's lemma

There is an interesting application of the spectral theorem in representation theory.

Definition 5.26. Given a group G, a unitary representation of G in a Hilbert space $\mathcal{H}$ is a group homomorphism $\pi\colon G \longrightarrow U(\mathcal{H})$ of G into the group $U(\mathcal{H})$ of unitary operators of $\mathcal{H}$.

We can readily distinguish an interesting property of certain unitary representations, called irreducibility. In summary, this is satisfied whenever the only invariant subspaces of $\mathcal{H}$ are (0) and $\mathcal{H}$ itself. For a subspace of $\mathcal{H}$ to be invariant with respect to π it means the following.

Definition 5.27. A subspace $\mathcal{L} \subset \mathcal{H}$ is said to be invariant if $\pi(g)\mathcal{L} \subset \mathcal{L}$ for every $g \in G$.

Observe that if $\mathcal{L}$ is invariant, so is $\mathcal{L}^\perp$: let $v \in \mathcal{L}^\perp$, that is, $\langle v, w\rangle = 0$ for every $w \in \mathcal{L}$. Then

$$\langle \pi(g)v, w\rangle = \langle v, \pi(g)^* w\rangle = \left\langle v, \pi(g^{-1})w\right\rangle,$$

if $w \in \mathcal{L}$ then $\pi(g^{-1})w \in \mathcal{L}$ for all g. This implies that $\langle \pi(g)v, w\rangle = 0$ for all $w \in \mathcal{L}$, hence $\pi(g)v \in \mathcal{L}^\perp$ for every $g \in G$ and all $v \in \mathcal{L}^\perp$.

Definition 5.28. A unitary representation $(\pi, \mathcal{H})$ is called irreducible if whenever $\mathcal{L} \subset \mathcal{H}$ is a closed invariant subspace, then we have either $\mathcal{L} = (0)$ or $\mathcal{L} = \mathcal{H}$.

Remark. Let G be countable and $\mathcal{H}$ of infinite dimension. Additionally, let $v \neq 0$ and consider the linear span of $\{\pi(g)v : g \in G\}$. This is an invariant, nonzero vector subspace of $\mathcal{H}$, and it cannot be equal to $\mathcal{H}$ since $\dim_{\mathbb{C}} \mathcal{H} = \aleph_1$.[1]

Let us assume $\mathcal{L} \subset \mathcal{H}$ is a closed $\pi(G)$-invariant subspace and let $\mathbf{P}\colon \mathcal{H} \longrightarrow \mathcal{L}$ be the orthogonal projection. If $v \in \mathcal{H}$, then it can be expressed as $v = v_1 + v_2$, where $v_1 \in \mathcal{L}$ and $v_2 \in \mathcal{L}^\perp$. Of course, $v_1 = \mathbf{P}(v)$. Now, for any $g \in G$,

$$\pi(g)v = \pi(g)v_1 + \pi(g)v_2,$$

[1] Standard exercise with Baire's theorem.

which since both $\mathcal{L}$ and $\mathcal{L}^\perp$ are $\pi(G)$-invariant, implies

$$\mathbf{P}(\pi(g)v) = \pi(g)v_1 = \pi(g)\mathbf{P}(v).$$

Therefore $\mathbf{P}\pi(g) = \pi(g)\mathbf{P}$. Noticing this, we maximize the set of operators that satisfy this and define

$$\mathrm{Int}(\pi) = \{T \in \mathcal{L}(\mathcal{H}) : \ T\pi(g) = \pi(g)T \quad \forall g \in G\}.$$

This set is a sub-C^*-algebra of $\mathcal{L}(\mathcal{H})$ containing $\{\lambda\,\mathrm{id}_\mathcal{H} : \ \lambda \in \mathbb{C}\}$. What we know from projections is that, if $\mathcal{L} \subset \mathcal{H}$ is a closed, invariant subspace and $\mathbf{P}\colon \mathcal{H} \longrightarrow \mathcal{L}$ is the orthogonal projection onto $\mathcal{L}$, then we know that $\mathbf{P} \in \mathrm{Int}(\pi)$. The key result that we will prove is the following.

Theorem 5.29 (*Schur's lemma*). The representation $(\pi, \mathcal{H})$ is irreducible if and only if $\mathrm{Int}(\pi) = \{\lambda\,\mathrm{id}_\mathcal{H} : \ \lambda \in \mathbb{C}\}$.

The proof of this lemma uses spectral theory and we will require the following lemma.

Lemma 5.30. Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator. Then $T = \lambda\,\mathrm{id}_\mathcal{H}$ if and only if $\mathrm{Sp}(T) = \{\lambda\}$.

Proof. The argument for the direct implication is as follows. Let $T = \lambda\,\mathrm{id}_\mathcal{H}$. Then $T - \lambda\,\mathrm{id}_\mathcal{H} = 0$, and therefore $\lambda \in \mathrm{Sp}(T)$, but for every $\mu \neq \lambda$, $T - \mu\,\mathrm{id}_\mathcal{H} = (\lambda - \mu)\,\mathrm{id}_\mathcal{H}$, which is invertible. Therefore the only element in the spectrum has to be λ.

For the converse, let E be the resolution of identity on $\mathrm{Sp}(T) = \{\lambda\}$ associated to T. Then $E(\{\lambda\}) = E(\mathrm{Sp}(T)) = \mathrm{id}_\mathcal{H}$. Now, observe that λ is an isolated point of $\mathrm{Sp}(T)$, hence $E(\{\lambda\})$ has to be the projection onto $\ker(T - \lambda\,\mathrm{id})$ (Lemma 5.11). Thus, $\ker(T - \lambda\,\mathrm{id}) = \mathcal{H}$, and $\lambda\,\mathrm{id} = T$. □

Now let us prove Schur's lemma.

Proof of Theorem 5.29. First assume that $(\pi, \mathcal{H})$ is not irreducible and $0 \subsetneq \mathcal{L} \subsetneq \mathcal{H}$ is a closed invariant subspace. Then we have shown that the orthogonal projection $\mathbf{P}$ onto $\mathcal{L}$ is in $\mathrm{Int}(\pi)$. This shows that $\mathrm{Int}(\pi) \supsetneq \{\lambda\,\mathrm{id}_\mathcal{H} : \ \lambda \in \mathbb{C}\}$.

Now assume that $\mathrm{Int}(\pi) \supsetneq \{\lambda\,\mathrm{id}_\mathcal{H} : \lambda \in \mathbb{C}\}$. Since $\mathrm{Int}(\pi)$ is a C^*-algebra, every operator T can be decomposed as $T = T_1 + iT_2$ with $T_1 = T_1^*$ and $T_2 = T_2^*$, with both $T_1, T_2 \in \mathrm{Int}(\pi)$. Indeed,

$$T_1 = \frac{T + T^*}{2}, \quad T_2 = \frac{T - T^*}{2i}.$$

So, if $T \in \mathrm{Int}(\pi)$, $T \notin \{\lambda\,\mathrm{id}_\mathcal{H} : \ \lambda \in \mathbb{C}\}$, then either T_1 or T_2 are not in $\{\lambda\,\mathrm{id}_\mathcal{H} : \ \lambda \in \mathbb{C}\}$. Thus there exists $S = S^*$ in $\mathrm{Int}(\pi)$ such that $S \notin \{\lambda\,\mathrm{id}_\mathcal{H} : \lambda \in \mathbb{C}\}$. In particular, S is normal and by Lemma 5.30, $\left|\mathrm{Sp}(T)\right| \geq 2$.

Let E be a resolution of identity on $\mathrm{Sp}(S)$ associated to S. Let λ_1, λ_2 be in $\mathrm{Sp}(S)$ and $V_i \ni \{\lambda_i\}$, $i = 1, 2$, open subsets with $V_1 \cap V_2 = \emptyset$. Since both V_1 and V_2 are open and nonempty, we have $E(V_i) \neq 0$ by Theorem 5.20 (iv), and we also have

$$E(V_1)E(V_2) = E(V_1 \cap V_2) = E(\emptyset) = 0.$$

Hence $E(V_1) \neq \mathrm{id}_\mathcal{H}$ and $E(V_2) \neq \mathrm{id}_\mathcal{H}$, so if $\mathcal{L}_1$ is the image of $E(V_1)$, which is closed and $0 \subsetneq \mathcal{L}_1 \subsetneq \mathcal{H}$, we can use the fact that any operator that commutes with S also has to commute with all the $E(\omega)$'s. But $\pi(g)S = S\pi(g)$ for all $g \in G$, and thus $\pi(g)E(V_1) = E(V_1)\pi(g)$, so $\mathcal{L}_1$ is $\pi(G)$-invariant. □

5.8 Positive operators and polar decomposition

We first study a special case of self-adjoint operators.

Definition 5.31. We say that $T \in \mathcal{L}(\mathcal{H})$ is positive, and write $T \geq 0$, if

$$\langle Tx, x\rangle \geq 0 \quad \forall x \in \mathcal{H}.$$

If T is positive, we have

$$\langle x, Tx\rangle = \overline{\langle Tx, x\rangle} = \langle Tx, x\rangle\,,$$

making it so that T is self-adjoint.

Theorem 5.32. Let $T \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent.

(i) T is positive.

(ii) T is self-adjoint and $\mathrm{Sp}(T) \subset [0, \infty)$.

Proof. Suppose T is positive, then T is self-adjoint, so $\mathrm{Sp}(T) \subset \mathbb{R}$. Now let $\lambda > 0$. For every $x \in \mathcal{H}$ we have the estimate

$$\lambda\|x\|^2 = \langle \lambda x, x\rangle \leq \langle \lambda x, x\rangle + \langle Tx, x\rangle = \langle(\lambda\,\mathrm{id} + T)x, x\rangle \leq \|(\lambda\,\mathrm{id} + T)x\|\|x\|.$$

Therefore, for $x \in \mathcal{H}$,

$$\|(\lambda\,\mathrm{id} + T)x\| \geq \lambda\|x\|.$$

Now T is self-adjoint and $\lambda \in \mathbb{R}$, so $T + \lambda\,\mathrm{id}$ is self-adjoint and consequently also normal. So by Proposition 5.8 we find that $T + \lambda\,\mathrm{id} = T - (-\lambda)\,\mathrm{id}$ is invertible. Therefore, $-\lambda \notin \mathrm{Sp}(T)$ for all $\lambda > 0$ meaning that $\mathrm{Sp}(T) \subset [0, \infty)$.

Now we prove the converse implication. Assume T is self-adjoint and $\mathrm{Sp}(T) \subset [0, \infty)$. Let E be the resolution of identity on $\mathrm{Sp}(T)$ so that

$$T = \int_{\mathrm{Sp}(T)} \lambda\,\mathrm{d}E(\lambda).$$

Then, for every $x \in \mathcal{H}$,

$$\langle Tx, x\rangle = \int_{\mathrm{Sp}(T)} \lambda\,\mathrm{d}E_{x,x}(\lambda) \geq 0,$$

since $\mathrm{Sp}(T) \subset [0, \infty)$, and $E_{x,x}$ is a positive regular Borel measure. □

The following theorem is about a single positive operator. Nevertheless, it uses the Guelfand isomorphism for abelian C^*-algebras.

Theorem 5.33. Every positive $T \in \mathcal{L}(\mathcal{H})$ has a unique positive square root S. If T is invertible, so is S.

Proof. Let E be the resolution of identity on $\mathrm{Sp}(T)$ with

$$T = \int_{\mathrm{Sp}(T)} \lambda \, \mathrm{d}E(\lambda).$$

Since $\mathrm{Sp}(T) \subset [0, \infty)$, the function $f(\lambda) = \sqrt{\lambda}$ is well defined and continuous on $\mathrm{Sp}(T)$. Hence,

$$S = f(T) = \int_{\mathrm{Sp}(T)} f(\lambda) \, \mathrm{d}E(\lambda) = \int_{\mathrm{Sp}(T)} \sqrt{\lambda} \, \mathrm{d}E(\lambda)$$

is self-adjoint, $S = S^*$, and $\mathrm{Sp}(S) \subset \mathrm{Sp}(T)$. We also have

$$S^2 = f(T)^2 = \int_{\mathrm{Sp}(T)} \lambda \, \mathrm{d}E = T.$$

Now we show that this is unique. Suppose S' satisfies the same properties as S. Let then E' be the resolution of the identity for S' that makes

$$S' = \int_{\mathrm{Sp}(S')} \lambda \, \mathrm{d}E'(\lambda).$$

Then $T = S'^2$ and thus for all $f \in C((0, \infty))$,

$$f(T) = f(S'^2) = \int_{\mathrm{Sp}(S')} f(\lambda^2) \, \mathrm{d}E'(\lambda) = \int_{\mathrm{Sp}(T)} f(\lambda) \, \mathrm{d}E(\lambda).$$

Therefore,

$$\int_{\mathrm{Sp}(T)} f(\lambda) \, \mathrm{d}E_{x,x}(\lambda) = \int_{\mathrm{Sp}(S')} f(\lambda^2) \, \mathrm{d}E'_{x,x}(\lambda),$$

which when replacing f by $f(\lambda) = g(\sqrt{\lambda})$ gives

$$\int_{\mathrm{Sp}(S')} g(\lambda) \, \mathrm{d}E'_{x,x}(\lambda) = \int_{\mathrm{Sp}(T)} g(\sqrt{\lambda}) \, \mathrm{d}E_{x,x}(\lambda).$$

This implies that $E'_{x,x}$ is uniquely determined by $E_{x,x}$, hence E' is uniquely determined by E, which implies uniqueness.

Finally, if T is invertible, then

$$\begin{aligned} (T^{-1}S)S &= T^{-1}S^2 = T^{-1}T = \mathrm{id} \\ S(T^{-1}S) &= T^{-1}(S^2) = T^{-1}T = \mathrm{id}\,. \end{aligned}$$

□

Now, if $T \in \mathcal{L}(\mathcal{H})$ is an arbitrary operator, consider T^*T, and observe

$$\langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2 \geq 0 \quad \forall x \in \mathcal{H}.$$

Therefore $T^*T \geq$, which hints at the following result.

Theorem 5.34. If $T \in \mathcal{L}(\mathcal{H})$ is an operator, the positive square root P of T^*T is the unique positive operator satisfying

$$\|Px\| = \|Tx\|, \quad \forall x \in \mathcal{H}.$$

Proof. If P is the positive square root of T^*T, we have

$$\|Px\|^2 = \langle P^2x, x\rangle = \langle T^*Tx, x\rangle = \|Tx\|^2$$

for all $x \in \mathcal{H}$. Conversely, if P' is a positive operator with $\|P'x\| = \|Tx\|$, then the computation above shows that

$$\langle P'^2x, x\rangle = \langle T^*Tx, x\rangle ,$$

hence $P'^2 = T^*T$ and thus $P' = P$ by the uniqueness part of Theorem 5.33. □

This allows us to generalize to Hilbert spaces a classical fact from linear algebra, namely that every invertible matrix $T \in GL(n, \mathbb{C})$ has a unique decomposition $T = UP$ where $U \in U(n)$ is unitary and P is hermitian, that is, $\overline{P}^T = P$ and positive definite. This is commonly called the ***polar decomposition*** of T.

Theorem 5.35. Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Then $T = UP$ with U unitary and $P \geq 0$. Moreover, this decomposition is unique.

Proof. Since T is invertible, so is T^* and hence T^*T. Therefore this last operator is positive and invertible, and thus has a positive square root P that is invertible as well. Let $U = TP^{-1}$. We compute

$$U^*U = (P*)^{-1}T^*TP^{-1} = P^{-1}T^*TP^{-1} = P^{-1}P^2P^{-1} = \text{id}\,.$$

Since U is invertible, we have $UU^* = \text{id}$.

For the uniqueness of this decomposition, take $T' = U'P'$ with U' unitary and $P' \geq 0$. Then

$$T^*T = (P')^*(U')^*U'P' = (P')^*P' = (P')^2.$$

Again, by the uniqueness part of Theorem 5.33 we get $P' = P$ and hence $U' = U$. □

6 Locally Compact Groups

This chapter puts together in an ad hoc way the basics of locally compact groups and Haar measure needed in the subsequent chapters. This material can be found in [EiWa] or [RaVa] chapter 1.

In this chapter we introduce topological groups and treat some examples with an emphasis on ***abelian topological groups***. In particular we discuss some details in the field of p-adics. Then we shortly discuss how the topology and group structure interact to produce some miraculous properties. The second part of the chapter will be devoted to introducing the Haar measure on a locally compact group, and establishing some basic properties of the convolution product.

6.1 Topological groups

Throughout this section we will always consider G to be a group.

Definition 6.1. A topology $\tau \subset \mathcal{P}(G)$ on the set G endows G with the structure of a topological group if the multiplication map

$$\begin{aligned} \cdot : G \times G &\longrightarrow G \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

and the inverse map

$$\begin{aligned} i : G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are both continuous.

In the above definition, $G \times G$ is endowed with the product topology of G. Let us draw some basic consequences from this first definition.

Remark. Let G be a topological group.

(i) The inverse map $i\colon G \longrightarrow G$ is continuous and in addition $i \circ i = \mathrm{id}$. Hence, i is a homeomorphism.

(ii) For $g \in G$, define the left translation by g as

$$\begin{aligned} L_g : G &\longrightarrow G \\ x &\longmapsto g \cdot x \end{aligned},$$

which is continuous by definition. Observe that $L_{g^{-1}} \circ L_g = L_g \circ L_{g^{-1}} = \mathrm{id}_G$, and hence L_g is a homeomorphism. So a topological group "looks" locally everywhere the same. Analogously, one defines right translation by $g \in G$ as

$$\begin{aligned} R_g \colon G &\longrightarrow G \\ x &\longmapsto x \cdot g \end{aligned}$$

and concludes that it is a homeomorphism as well.

(iii) Let $\varphi \colon G_1 \longrightarrow G_2$ be a homomorphism where G_1, G_2 are both topological groups. Assume φ to be continuous at $e \in G_1$, the identity in G_1. Then, given $g \in G$ arbitrary, $\varphi \circ L_{g^{-1}}$ is continuous at g since $L_{g^{-1}}(g) = e$. But

$$\varphi \circ L_{g^{-1}}(h) = \varphi(g^{-1})\varphi(h),$$

which can also be written as

$$L_{\varphi(g)} \circ \varphi \circ L_{g^{-1}} = \varphi,$$

given that φ is a homomorphism, implying that φ is continuous at $g \in G$. Since $g \in G$ is arbitrary, we have established that a homomorphism is continuous if and only if it is continuous at $e \in G_1$.

(iv) Let $H < G$ be a subgroup of G. Then, with the induced topology, H is a topological group.

Now we turn towards some examples.

Example 6.2 (*Topological groups*).

(i) Any group G endowed with the discrete topology is a topological group.

(ii) The space $\mathbb{R}^n$, equipped with the regular addition and the Euclidean topology is an abelian topological group.

(iii) If A is a unital Banach algebra, the group $G(A)$ of invertible elements with the topology induced from A is a topological group (recall Lemma 2.8).

(iv) The additive groups $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$, as well as the multiplicative groups $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$ of the fields $\mathbb{R}$ and $\mathbb{C}$ are abelian topological groups.

Observe that examples (i), (ii) and (iv) are locally compact Hausdorff, while $G(A)$ is locally compact if and only if A is finite dimensional (which we will leave for the reader to prove as an easy exercise). In this context the following examples are a special case of (iii)

(v) The groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are locally compact Hausdorff groups.

The cartesian product leads to a wealth of examples.

(vi) Let G_α, where $\alpha \in A$ be a family of topological groups. Endow the product $G = \prod_{\alpha \in A} G_\alpha$ with the componentwise product and the product topology. Then G is a topological group. It is compact if G_α is compact for every $\alpha \in A$. For instance, if we endow $\mathbb{Z}/2\mathbb{Z}$ with the discrete topology, then $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is a compact group. It is an instructive exercise to show that G is locally compact Hausdorff if and only if all the G_α's are locally compact Hausdorff and all but finitely many G_α's are compact.

There is, however, a more general source of locally compact topological groups.

(vii) Let (X, d) be a metric space such that all closed balls of finite radius are compact. Then the group $\mathrm{Is}(X)$ of isometries of (X, d) with compact open topology is locally compact Hausdorff.

(viii) The ring $\mathbb{Z}_p$ of p-adic integers is a locally compact Hausdorff topological group. For every $n \geq 1$, let $A_n := \mathbb{Z}/p^n\mathbb{Z}$ be the ring of integers modulo p^n. Given $x \in A_n$, its reduction modulo p^{n-1} is well defined and leads to a surjective ring homomorphism

$$\begin{aligned} \phi_n \colon A_n &\longrightarrow A_{n-1} \\ x &\longmapsto x \mod p^{n-1}. \end{aligned}$$

We obtain a sequence of rings with morphisms connecting them:

$$A_1 \xleftarrow{\phi_2} A_2 \xleftarrow{\phi_3} A_3 \longleftarrow \ldots \longleftarrow A_{n-1} \xleftarrow{\phi_n} A_n \longleftarrow \ldots$$

The ring $\mathbb{Z}_p$ of p-adic integers is then the projective limit of the system (A_n, ϕ_n) defined above. By definition, $\mathbb{Z}_p$ is the subring of $\prod_{n \geq 1} A_n$ given by

$$\mathbb{Z}_p = \left\{ (x_1, x_2, \ldots) \in \prod_{n \geq 1} A_n : \ \phi_n(x_n) = x_{n-1} \quad \forall n \geq 2 \right\}.$$

More precisely, coordinatewise addition and multiplication on $\prod_{n \geq 1} A_n$ makes it a ring, and since the ϕ_n are homomorphisms, $\mathbb{Z}_p$ is a subring. Then $\mathbb{Z}$ injects into $\mathbb{Z}_p$ via

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z}_p \\ x &\longmapsto (x \mod p, \ x \mod p^2, \ldots) \end{aligned}$$

and we identify it with a subring of $\mathbb{Z}_p$.

Now, endow A_n with the discrete topology. Then $\prod_{n \geq 1} A_n$ is compact Hausdorff, and $\mathbb{Z}_p$ being defined by closed conditions is hence compact Hausdorff. Both operations of addition and multiplication are continuous, and so is $x \mapsto -x$. In particular, $(\mathbb{Z}_p, +)$ is an abelian compact Hausdorff group. Additionally, $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.

We leave it to the reader to check the following fact.

Exercise

It is well known that if $p \equiv 1 \mod 4$, then -1 is a square in $\mathbb{Z}/p\mathbb{Z}$, that is, $x^2 + 1 = 0$ has a solution in $\mathbb{Z}/p\mathbb{Z}$. Use this, together with some elementary computations to inductively construct a sequence in $x_n \in \mathbb{Z}/p^n\mathbb{Z}$ with

(1) $x_n^2 + 1 = 0$ in $\mathbb{Z}/p^n\mathbb{Z}$.

(2) $\phi(x_n) = x_{n-1}$.

Thus $x^2+1=0$ has a solution in $\mathbb{Z}_p$.

One important part of the study of $\mathbb{Z}_p$ is its relation to the A_n's. In fact, associating to $x = (x_1, x_2, ...) \in \mathbb{Z}_p$ its n-th component $x_n \in A_n$ defines continuous ring homomorphisms

$$\begin{aligned} \varepsilon_n \colon \mathbb{Z}_p &\longrightarrow A_n \\ x &\longmapsto x_n \end{aligned}$$

whose kernel is $p^n\mathbb{Z}_p$. The latter is hence an open subgroup of $\mathbb{Z}_p$ and since $\bigcap_{n\geq 1} p^n\mathbb{Z}_p = (0)$, they form a fundamental system of neighbourhoods of (0). We thus have

(1) $x \in \mathbb{Z}_p$ is invertible if and only if $x \notin p\mathbb{Z}_p$.

(2) If $U \subset \mathbb{Z}_p$ denotes the group of invertible elements, then every $x \in \mathbb{Z}_p \setminus \{0\}$ can be written uniquely as $x = p^n u$, with $n \geq 0$ and $u \in U$.

The first property follows from the analogous statement for A_n, whereas the second follows from $\bigcap_{n\geq 1} p^n\mathbb{Z}_p = \{0\}$.

(i) The field $\mathbb{Q}_p$. Observe that $\mathbb{Z}_p$ is an integral domain. Indeed, if $xy = 0$ and $x, y \neq 0$, write $x = p^n u$ and $y = p^m u'$. Then $xy = p^{n+m}uu' = 0$, meaning that $p^{n+m} = 0$, which is nonsense. One deduces easily that $\mathbb{Q}_p$ can be identified with $\mathbb{Z}_p[p^{-1}]$. In fact, every $x \in \mathbb{Q}_p^\times$ can be written uniquely as $x = p^n u$, for $n \in \mathbb{Z}$ and $u \in U$. If we set $v(x) := n \in \mathbb{Z}$, we obtain a so called ***valuation***, that is

$$v\colon \mathbb{Q}_p \longrightarrow \mathbb{Z} \cup \{\infty\}$$

where we put $v(0) = \infty$, satisfying

$$\begin{aligned} v(xy) &= v(x) + v(y) \\ v(x+y) &\geq \min\{v(x), v(y)\}\,. \end{aligned}$$

It follows that $d(x,y) := e^{-v(x-y)}$ defines a distance on $\mathbb{Q}_p$: it induces the given topology on $\mathbb{Z}_p$. One concludes that $\mathbb{Q}_p$ is locally compact and contains $\mathbb{Z}_p$ as an open subring. Additionally, $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.

Having discussed these examples, we illustrate why the last two are important. In particular, they play major roles in local approaches to algebraic number theory:

(1) Assume K is a non-discrete, locally compact Hausdorff field with char $K = 0$. Then K is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$ for some p.

(2) If $i\colon \mathbb{Q} \longrightarrow K$ is an injection with dense image and K is locally compact and non-discrete, then $K = \mathbb{R}$ or $\mathbb{Q}_p$ for some p.

For more on this, see [We] chapter 1 or [RaVa] chapter 4.

6.2 Properties of topological groups

Recall that a topological space is connected if it cannot be written as the union of two disjoint, non-empty, open subsets. Recall as well that the closure of a connected set is connected and the continuous image of a connected set is as well connected. Finally, given a topological space X, the relation $x \sim y$ if $\{x, y\}$ is contained in a connected subset of X is an equivalence relation, and its equivalence classes are called connected components. We have the following proposition.

Proposition 6.3. Let G be a topological group. Then we have the following.

(i) If $H \leq G$ is a subgroup, then so is its closure $\bar{H} \leq G$.

(ii) If $H < G$ is an open subgroup then it is closed.

(iii) The connected component G_0 of $e \in G$ is a closed normal subgroup.

(iv) If G is connected and $V \ni e$ is a neighborhood of e, then $V \cup V^{-1}$ generates G, i.e.,

$$G = \bigcup_{n \geq 1} (V \cup V^{-1})^n.$$

Let us introduce some notation before moving onto the proof. Given subsets $A, B \subset G$, we denote

$$A \cdot B := \{a \cdot b : \; a \in A, \; b \in B\},$$
$$A^{-1} = \left\{a^{-1} : \; a \in A\right\},$$
$$A^n = A \cdot A^{n-1}, \quad n \geq 2.$$

Proof of Proposition 6.3.

(i) Recall that given $f : X \longrightarrow Y$ continuous and $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$. Applying this to the multiplication and inversion maps we get

$$m(\overline{H} \times \overline{H}) = m(\overline{H \times H}) \subset \overline{m(H \times H)} = \overline{H}$$

and

$$i(\overline{H}) \subset \overline{i(H)} = \overline{H},$$

which proves (i).

(ii) Let $R \subset G$ be a complete set of representatives for G/H with $e \in R$, i.e.

$$G = \bigsqcup_{x \in R} x \cdot H = H \sqcup \bigsqcup_{x \in \mathbb{R} \setminus \{e\}} L_x(H).$$

Since H is open and $L_x : G \longrightarrow G$ is a homeomorphism, so are $L_x(H)$ and $\bigsqcup_{x \in R \setminus \{e\}} L_x(H)$, which implies that H is closed.

(iii) Notice first that connected components are always closed. Thus, the subset G_0 is closed. Now $m(G_0 \times G_0) \subset G$ and $i(G_0) \subset G$ are connected subsets containing $e \in G$. Hence they are contained in G_0, which shows that G_0 is a subgroup. Finally, given $g \in G$, and observing that the map $x \mapsto gxg^{-1}$ is continuous, we have that

$$\{gxg^{-1} : x \in G_0\}$$

is a connected subset of G containing e, hence it is contained in G_0. This shows that G_0 is normal in G.

(iv) Observe that $H := \bigcup_{n\geq 1}(V \cup V^{-1})^n$ is a subgroup of G. Now let $e \in U \subset V$ be an open subset of G. Then $U \subset H$ and since H is a group, we have

$$L_h(U) \subset H, \quad \forall h \in H.$$

Since $L_h(U) \ni h$ is an open neighborhood of h, the set H is a neighborhood of each of its points, hence it is open. By (ii), it is closed, and since G is connected and H is nonempty, we deduce $H = G$. □

Remark. According to (iii) we can write $G = \bigsqcup xG_0$, where the union is over a complete set of representatives of G/G_0. Hence the set $\pi_0(G)$ of connected components of G acquires, via its identification with G/G_0, a group structure. It is an instructive exercise to compute $\pi_0(G)$ in each of the examples at the beginning of the chapter.

6.3 Haar measure

Let now G be a locally compact Hausdorff group and let $C_{00}(G)$ denote the $\mathbb{C}$-vector space of continuous compactly supported functions. The most important fact about this class of groups is the existence and uniqueness of the Haar measure.

Given a map $F : G \to X$, define the action

$$\begin{aligned} \lambda(g)F : G &\longrightarrow X \\ x &\longmapsto F(g^{-1}x). \end{aligned}$$

Clearly if $F \in C_{00}(G)$ then $\lambda(g)F \in C_{00}(G)$, and $\lambda : G \mapsto GL(C_{00}(G))$, with $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$. Thus, λ is a group homomorphism. The fundamental result then comes presented as the following theorem.

Theorem 6.4 (*Existence and uniqueness of the Haar measure*). Let G be a locally compact Hausdorff group. Then there exists a non-zero, positive linear functional $\Lambda : C_{00}(G) \to \mathbb{C}$ that is invariant under left translations, i.e.

$$\Lambda(\lambda(g)f) = \Lambda(f), \quad \forall f \in C_{00}(G), \forall g \in G.$$

Moreover, this functional is called a ***left Haar functional*** and is unique up to a strictly positive scalar multiple.

Using Riesz's representation theorem, one obtains the following equivalent formulation.

Corollary 6.5. There exists, up to a strictly positive scalar multiple, a unique non-zero, positive regular Borel measure μ on G such that for every measurable set $E \subset G$, and every $g \in G$, $\mu(g \cdot E) = \mu(E)$.

This measure is referred to as the ***left Haar measure***. In case G is abelian, we have $L_g = R_g$ for every $g \in G$, and a left Haar measure is simply a ***Haar measure***. We will not give a proof of Haar's theorem in these notes. For a proof, see [EiWa] 10.1.

Example 6.6.

(i) The Lebesgue measure on $(\mathbb{R}, +)$ is the unique positive, regular Borel on measure on $\mathbb{R}$ with $\mu([a,b]) = b - a$ for $a \leq b$. It is a Haar measure for $(\mathbb{R}, +)$, and so is the Lebesgue measure in $\mathbb{R}^n$ for $(\mathbb{R}^n, +)$.

(ii) A Haar measure on $(\mathbb{R}^\times, \cdot)$ is given by $\mathrm{d}\mu(x) = \mathrm{d}x/|x|$.

(iii) Let G be discrete, then
$$\mu(E) = \operatorname{card} E, \quad E \subset G$$
is a left Haar measure.

The uniqueness statement in Haar's theorem is probably more important, since it immediately gives additional structure. Let $\operatorname{Aut}(G)$ denote the group of continuous automorphisms of G.

Corollary 6.7. There is a well defined group homomorphism $\operatorname{mod}_G \colon \operatorname{Aut}(G) \longrightarrow \mathbb{R}_{>0}$ into the multiplicative group $(\mathbb{R}_{>0}, \cdot)$ such that for any left Haar functional Λ,
$$\Lambda(f \circ \alpha^{-1}) = \operatorname{mod}_G(\alpha)\Lambda(f), \quad \forall f \in C_{00}(G), \alpha \in \operatorname{Aut}(G).$$

Proof. Observe that $f \mapsto f \circ \alpha^{-1}$ is a linear map on $C_{00}(G)$ preserving positivity. In addition, we have, for $\alpha \in \operatorname{Aut}_G$, $f \in C_{00}(G)$ and $g, x \in G$,
$$(\lambda(g)f)\left(\alpha^{-1}(x)\right) = f\left(g^{-1}\alpha^{-1}(x)\right) = f\left(\alpha^{-1}\left(\alpha(g)^{-1}x\right)\right) = \lambda(\alpha(g))\left(f \circ \alpha^{-1}\right)(x).$$
As a consequence, if we define the functional $\Lambda_\alpha(f) := \Lambda(f \circ \alpha^{-1})$, then Λ_α is a nonzero, positive functional with
$$\Lambda_\alpha(\lambda(g)f) = \Lambda\big((\lambda(g)f) \circ \alpha^{-1}\big) = \Lambda\big(\lambda(\alpha(g))\big(f \circ \alpha^{-1}\big)\big) = \Lambda\big(f \circ \alpha^{-1}\big) = \Lambda_\alpha(f).$$
Hence, Λ_α is a Haar functional. By uniqueness, there is a constant $c_\Lambda(\alpha) > 0$ for which $\Lambda_\alpha = c_\Lambda(\alpha)\Lambda$. One then verifies easily that $c_\Lambda(\alpha)$ is independent of the choice of Λ, and defines
$$\operatorname{mod}_G(\alpha) = c_\Lambda(\alpha).$$
Then, by choosing α_1, α_2 and associating two functionals Λ_{α_i}, $i = 1, 2$, we examine the induced $\Lambda_{\alpha_1 \circ \alpha_2}$, and by uniqueness again, we reach the conclusion that
$$\operatorname{mod}_G(\alpha_1 \circ \alpha_2) = \operatorname{mod}_G(\alpha_1)\operatorname{mod}_G(\alpha_2). \qquad \square$$

Example 6.8.

(i) Let $G = (\mathbb{R}^n, +)$. Then, it is an exercise to check that $\operatorname{Aut}(G) = GL(n, \mathbb{R})$, and
$$\operatorname{mod}_G(\alpha) = |\det \alpha|.$$
Indeed, if $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^n$, $\mathcal{L}(\alpha([0,1]^n)) = |\det \alpha| \cdot \mathcal{L}([0,1]^n)$.

(ii) Let K be a field with a locally compact topology, making $(K,+)$ and $(K^\times,\cdot)$ locally compact Hausdorff groups (for instance, $\mathbb{R},\mathbb{C},\mathbb{Q}_p$ for p prime). Consider $K^\times = GL(1,K) \hookrightarrow \mathrm{Aut}(K,+)$, that gives us a canonical homomorphism $K^\times \longrightarrow \mathbb{R}_{>0}$ which, if μ is any Haar measure on $(K,+)$, verifies

$$\mu(y\cdot E) = \mathrm{mod}_k(y)\cdot\mu(E) \quad \forall y\in K^\times, \forall E\subset k \text{ measurable}.$$

If K is not discrete, one can show that $y\mapsto \mathrm{mod}_K(y)$ is not identically 1, and behaves somewhat like an absolute value, that is, there exists a constant $c\geq 0$ with

$$\mathrm{mod}_K(y_1+y_2)\leq c\max\{\mathrm{mod}_K(y_1),\mathrm{mod}_K(y_2)\}.$$

This is the starting point of the classification of non-discrete locally compact fields. It is a very non-trivial result.

(iii) In the case of $(\mathbb{Q}_p,+)$, one can determine $\mathrm{mod}_{\mathbb{Q}_p}$. Indeed, we may assume $y\in\mathbb{Z}_p$, since $\mathrm{mod}(y^{-1})=\mathrm{mod}(y)^{-1}$. Write $y=p^n u$ for some $n\geq 0$ and $u\in U$ invertible in $\mathbb{Z}_p$. Now $\mathbb{Z}_p$ is open in $\mathbb{Q}_p$ and it is compact, so if μ is a Haar measure on $\mathbb{Q}_p$, then $\mu(\mathbb{Z}_p)>0$. On the one hand, we have $\mu(y\cdot\mathbb{Z}_p)=\mathrm{mod}(y)\mu(\mathbb{Z}_p)$. On the other hand, $y\mathbb{Z}_p=p^n\mathbb{Z}_p$, and the latter is the kernel of the homomorphism

$$\varepsilon_n\colon \mathbb{Z}_p\longrightarrow A_n=\mathbb{Z}/p^n\mathbb{Z}.$$

Let then R be a complete set of representatives of A_n, so $|R|=p^n$ and $\mathbb{Z}_p=\bigsqcup_{r\in R}(r+p^n\mathbb{Z}_p)$. Using additivity and translation invariance of μ, this implies

$$\mu(\mathbb{Z}_p)=\sum_{r\in R}\mu(r+p^n\mathbb{Z}_p)=p^n\mu(p^n\mathbb{Z}_p).$$

Hence $\mu(p^n\mathbb{Z}_p)=p^{-n}\mu(\mathbb{Z}_p)$ and $\mathrm{mod}(y)=p^{-n}$.

Remark. In the preceding example we have used the fact that if μ is a left Haar measure on G, then for any non-empty, open set $V\subset G$, its measure $\mu(V)$ is positive. Indeed, let $f\in C_{00}(G)$ with $f\geq 0$ be such that

$$\int_G f(x)\,\mathrm{d}\mu(x)>0.$$

Since f is supported on a compact set, we may find $x_1,\ldots,x_n\in G$ such that

$$\mathrm{supp}\, f\subset\bigcup_{i=1}^n x_i\cdot V.$$

If $M=\max\{f(x):\ x\in G\}$, then we have

$$f(x)\leq M\sum_{i=1}^n \chi_{x_i\cdot V}(x)\quad \forall x\in G,$$

and thus

$$0<\int_G f(x)\,\mathrm{d}\mu(x)\leq\int_G M\sum_{i=1}^n\chi_{x_i\cdot V}\,\mathrm{d}\mu(x)=M\sum_{i=1}^n\mu(x_iV)=Mn\mu(V),$$

which implies $\mu(V)>0$.

We will explore one additional aspect of Haar's theory. Notice what happens when we apply the previous corollary to a special class of automorphisms of G, the inner automorphisms

$$\alpha_g(x) := gxg^{-1}, \quad x \in G.$$

If μ is a left Haar measure, the equality in the corollary reads

$$\int_G f(g^{-1}xg)\,\mathrm{d}\mu(x) = \mathrm{mod}_G(\alpha_g) \int_C f(x)\,\mathrm{d}\mu(x),$$

which, taking into account the left invariance of μ implies

$$\int_G f(xg)\,\mathrm{d}\mu(x) = \mathrm{mod}_G(\alpha_g) \int_G f(x)\,\mathrm{d}\mu(x).$$

This leads to the following definition.

Definition 6.9. Given a locally compact Hausdorff topological group G, we define its modular function by $\Delta_G(g) := \mathrm{mod}_G(\alpha_{g^{-1}})$, so that

$$\int_G f(xg^{-1})\,\mathrm{d}\mu(x) = \Delta_G(g) \int_G f(x)\,\mathrm{d}\mu(x).$$

With this definition, notice that if G is abelian, then $\Delta_G \equiv 1$. This leads to the following proposition.

Proposition 6.10. Let G be locally compact Hausdorff and abelian, and let μ be a left Haar measure. Then

$$\int_G f(x^{-1})\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}\mu(x), \quad \forall f \in C_{00}(G).$$

By density, this extends to $f \in L^1(G)$.

In full generality, the statement can be extended to a locally compact Hausdorff group G, not requiring that it is abelian. The statement then reads

$$\int_G f(x^{-1})\Delta_G(x^{-1})\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}\mu(x), \quad \forall f \in C_{00}(G).$$

Proof of Proposition 6.10. Define the linear functional

$$\begin{aligned} I\colon C_{00}(G) &\longrightarrow \mathbb{C} \\ f &\longmapsto \textstyle\int_G f(x^{-1})\,\mathrm{d}\mu(x). \end{aligned}$$

Then we can compute

$$\begin{aligned} I(\lambda(y)f) &= \int_G (\lambda(y)f)(x^{-1})\,\mathrm{d}\mu(x) = \int_G f(y^{-1}x^{-1})\,\mathrm{d}\mu(x) \\ &= \int_G f((yx)^{-1})\,\mathrm{d}\mu(x) = \int_G f(x^{-1})\,\mathrm{d}\mu(x) = I(f), \end{aligned}$$

where the invariance of the measure was used in the last equality. It follows that $I(f)$ is a left Haar functional and there exists $c > 0$ such that

$$\int_G f(x^{-1})\,\mathrm{d}x = c \cdot \int_G f(x)\,\mathrm{d}\mu(x), \quad \forall f \in C_{00}(G).$$

It remains to show that $c = 1$. For this, pick $f \in C_{00}(G)$, not identically zero, such that $f \geq 0$ and $\int_G f(x)\,\mathrm{d}\mu(x) > 0$. It's support is then a set of positive measure, $\mu(\mathrm{supp}(f)) > 0$. Define now $g(x) := f(x) + f(x^{-1})$, and notice it satisfies $g(x^{-1}) = g(x)$. A simple computation shows

$$\int_G g(x)\,\mathrm{d}\mu(x) = c\int_G g(x^{-1})\,\mathrm{d}\mu(x) = c\int_G g(x)\,\mathrm{d}\mu(x),$$

which implies that $c = 1$ since g is supported on a set of positive measure as so is f. □

6.4 The convolution product

For a locally compact Hausdorff group G and a left Haar measure μ, we denote by $L^p(G) := L^p(G,\mu)$ the usual L^p spaces associated to μ. We are going to define the convolution product of two functions on G and show that $L^1(G)$ is an involutive abelian Banach algebra. The results stated hold for all locally compact Hausdorff, abelian groups. The proofs however will be performed for σ-compact groups, i.e. groups that are a countable union of compact subsets. In this context we can use the classical version of Fubini's theorem as the measure space (G,μ) is σ-finite. We will later use that for $1 \leq p < \infty$, $L^q(G)$ is the dual Banach space of $L^p(G)$, where $1/p + 1/q = 1$. In particular, $L^\infty(G)$ is the dual of $L^1(G)$.

Let $f, g \in L^1(G)$ and apply Fubini's theorem to the positive measurable function

$$\begin{aligned} G \times G &\longrightarrow [0,\infty] \\ (x,y) &\longmapsto \left|f(xy)g(y^{-1})\right|. \end{aligned}$$

We get

$$\begin{aligned} \int_G \mathrm{d}\mu(x)\int_G \mathrm{d}\mu(y)\,\left|f(xy)g(y^{-1})\right| &= \int_G \mathrm{d}\mu(x)\,\left|g(y^{-1})\right|\int_G \mathrm{d}\mu(y)\,|f(xy)| \\ &= \int_G \mathrm{d}\mu(x)\,\left|g(y^{-1})\right|\int_G \mathrm{d}\mu(y)\,|f(x)| \\ &= \int_G \mathrm{d}\mu(x)\,|g(y)|\int_G \mathrm{d}\mu(y)\,|f(x)| = \|g\|_1\|f\|_1 < \infty, \end{aligned}$$

using the invariance of the measure several times. By Lebesgue's theorem,

$$f * g(x) := \int_G \mathrm{d}\mu(y) f(xy)g(y^{-1})$$

is finite a.e. and measurable, and $\|f * g\|_1 \leq \|f\|_1\|g\|_1$. Let also $f^*(x) = \overline{f(x^{-1})}$.

Theorem 6.11. The space $L^1(G)$, endowed with the convolution product and $f \mapsto f^*$ is an involutive Banach algebra.

Exercise

In order to prove the theorem, it remains to show that the convolution product is associative and the involution satisfies the required properties.

We now turn our aim towards finding a translation of the classical theory of Fourier analysis into the context we are working in. For this we will need some continuity properties of the convolution product. We begin with a definition.

Definition 6.12. Let (X, d) be a metric space and G a locally compact Hausdorff group. A function $f\colon G \longrightarrow X$ is left uniformly continuous, if for all $\varepsilon > 0$, there exists a neighborhood $V \ni e$ of e such that $\forall y^{-1}x \in V$, $d(f(x), f(y)) < \varepsilon$.

The terminology comes from the fact that if $y^{-1}x \in V$, then $(gy)^{-1}(gx) = y^{-1}x \in V$ for every $g \in G$. Hence, $d(f(gx), f(gy)) < \varepsilon$ for all $g \in G$.

Lemma 6.13. The following statements hold.

(i) For all $f \in C_{00}(G)$, the function

$$\begin{aligned} G &\longrightarrow C_{00}(G) \\ x &\longmapsto \lambda(x)f \end{aligned}$$

is left uniformly continuous with respect to the norm $\|\cdot\|_\infty$.

(ii) For all $1 \le p < \infty$, $f \in L^p(G)$,

$$\begin{aligned} G &\longrightarrow L^p(G) \\ x &\longmapsto \lambda(x)f \end{aligned}$$

is left uniformly continuous with respect to the norm $\|\cdot\|_p$.

Proof. Step 1. For (i), let $f \in C_{00}(G)$. Since $\|\cdot\|_\infty$ is left translation invariant, we have

$$\|\lambda(x)f - \lambda(y)f\|_\infty = \|\lambda(y^{-1}x)f - f\|_\infty.$$

Since $(\lambda(y^{-1}x)f)(g) = f(x^{-1}yg)$ it suffices to show that $\forall \varepsilon > 0$, $\exists W \ni e$ open such that $|f(zg) - f(g)| < \varepsilon$ for every $g \in G, z \in W$.

We begin by constructing the set W. Let $K = \operatorname{supp}(f)$ and let $V_0 = V_0^{-1} \ni e$ open with $\overline{V_0}$ compact, so that $\overline{V_0} \cdot K$ is compact. Let $\varepsilon > 0$. For every $x \in G$, let $V_x \ni e$ open with $V_x \subset V_0$ such that

$$|f(zx) - f(x)| < \varepsilon/2, \quad \forall z \in V_x.$$

By continuity of the multiplication, choose $U_x \ni e$ open such that $U_x^2 \subset V_x$. Given that $\overline{V_0} \cdot K$ is compact, we can extract a finite covering $\{U_{x_1} \cdot x_i\}_{i=1}^n$ of $\overline{V_0} \cdot K$ and consider the open set

$$W = \bigcap_{i=1}^n U_{x_i} \subset V_0.$$

Let us now verify that $|f(zg) - f(g)| < \varepsilon$ for every $z \in W, g \in G$.

(1) If $g \notin \overline{V_0} \cdot K$ then if $z \in W \subset V_0 = V_0^{-1}$ and hence $zg \notin K$. Indeed otherwise, this would imply $g \in z^{-1}K \subset V_0 K \subset \overline{V_0}K$, contradiction. Hence $f(g) = 0$ and $f(zg) = 0$.

(2) If $g \in \overline{V_0} \cdot K$, let $1 \le i \le n$ be such that $g \in U_{x_i} \cdot x_i$. Then since $z \in W \subset U_{x_i}$ we get $zg \in U_{x_i} \cdot U_{x_i} \cdot x_i \subset V_{x_i} \cdot x_i$. Therefore

$$|f(zg) - f(g)| \le |f(zg) - f(x_i)| + |f(x_i) - f(y)| < \varepsilon.$$

Step 2. For (ii), let $1 \le p < \infty$, and recall that $C_{00}(G)$ is dense in $L^p(G)$ with respect to the norm $\|\cdot\|_p$. Let then $f \in L^p(G)$ and $\varepsilon > 0$. By density, pick $g \in C_{00}(G)$ with $\|f - g\|_p < \varepsilon$. We compute

$$\begin{aligned}\|\lambda(x)f-\lambda(y)f\|_p &\le \|\lambda(x)(f-g)\|_p+\|\lambda(x)g-\lambda(y)g\|_p+\|\lambda(y)(g-f)\|_p\\ &=2\|f-g\|_p+\|\lambda(y^{-1}x)g-g\|_p<2\varepsilon+\|\lambda(y^{-1}x)g-g\|_p.\end{aligned}$$

Let $V_0=V_0^{-1}\ni e$ open, with $\overline{V_0}$ compact. We may assume that $y^{-1}x\in V_0$. Observe that $\forall t\notin\overline{V_0}\,\mathrm{supp}(g)=C$,

$$\lambda(y^{-1}x)g(t)-g(t)=0.$$

Therefore,

$$\|\lambda(y^{-1}x)g-g\|_p=\left\{\int_C\left|\lambda(y^{-1}x)g(t)-g(t)\right|^p\mathrm{d}\mu(x)\right\}^{1/p}\le\|\lambda(y^{-1}x)g-g\|_\infty\mu(C)^{1/p}.$$

Now choose $e\in W\subset V_0$ open such that $\|\lambda(y^{-1}x)g-g\|_\infty<\varepsilon/\mu(C)^{1/p}$ for every $y^{-1}x\in W$. We conclude that

$$\|\lambda(x)f-\lambda(y)f\|_p<3\varepsilon.$$

□

To finish this chapter, we conclude with some continuity properties of the convolution product in $L^1(G)$. We will make use of these in chapter 7 when developing Fourier analysis on locally compact Hausdorff, abelian groups.

Theorem 6.14. Let G be a locally compact Hausdorff, abelian group.

(i) If $f\in L^1(G)$ and $g\in L^\infty(G)$ then $f*g$ is bounded and uniformly continuous.

(ii) If $f,g\in C_{00}(G)$ then $f*g\in C_{00}(G)$ and $\mathrm{supp}(f*g)\subset\mathrm{supp}(f)\cdot\mathrm{supp}(g)$.

(iii) For $1<p<\infty$ and $f\in L^p(G)$, $g\in L^q(G)$, we have $f*g\in C_0(G)$.

Proof.

(i) We have

$$|f*g(x)|=\left|\int_G f(xy)g(y^{-1})\,\mathrm{d}\mu(y)\right|\le\|f\|_1\|g\|_\infty.$$

For $x,z\in G$, it holds

$$\begin{aligned}|f*g(x)-f*g(z)|&\le\int_G|f(xy)-f(zy)|\left|g(y^{-1})\right|\mathrm{d}\mu(y)\\ &\le\|\lambda(x^{-1})f-\lambda(z^{-1})f\|_1\|g\|_\infty,\end{aligned}$$

which together with the previous lemma shows (i).

(ii) We already know that $f*g$ is continuous by (i). If $f*g(x)\neq0$ and there is $y\in G$ with $f(xy)\neq0$ and $g(y^{-1})\neq0$, then $xy\in\mathrm{supp}\,f$ and $y^{-1}\in\mathrm{supp}\,g$. Hence $x\in(\mathrm{supp}\,f)y^{-1}$, meaning that $x\in\mathrm{supp}\,f\cdot\mathrm{supp}\,g$

(iii) Observe first that if $f\in L^p(G)$ and $g\in L^q(G)$, then, since G is abelian, $y\mapsto g(y^{-1})$ is in $L^q(G)$. It follows then from Hölder's inequality that for every $x\in G$, the map $y\mapsto f(xy)g(y^{-1})$ is in $L^1(G)$ and $f*g$ is defined.

Next we use that since $1 < p < \infty$, $C_{00}(G)$ is dense in $L^p(G)$ and $L^q(G)$. Choose approximating sequences f_n and g_n in $C_{00}(G)$ that converge to f and g in $L^p(G)$ and $L^q(G)$ respectively. Then,

$$\begin{aligned} |f * g(x) - f_n * g_n(x)| &\leq |(f - f_n) * g_n(x)| + |f_n * (g_n - g)(x)| \\ &\leq \|f - f_n\|_p \|g_n\|_q + \|f_n\|_p \|g_n - g\|_q, \end{aligned}$$

and hence $\|f * g - f_n * g_n\|_\infty$ converges to 0 as $n \to \infty$. This implies that $f * g \in C_0(G)$ since $f_n * g_n \in C_{00}(G)$.

□

7 The Fourier transform

The material of this chapter is mainly taken from [Ru1] 1.1 and 1.2.

Let G be locally compact abelian Hausdorff. We have seen in Theorem 6.11 that $L^1(G)$ with the convolution product and involution $f^*(x) = \overline{f(x^{-1})}$ is an involutive abelian Banach algebra.

Our first task is to identify the Guelfand spectrum of $L^1(G)$. This will turn out to be described by the continuous characters of G. Recall that $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. A character of G is a homomorphism $\chi\colon G \longrightarrow \mathbb{T}$. We denote by $\widehat{G}$ the set of all continuous characters of G. In fact, $\widehat{G}$ is in a natural way an abelian group: if

$$\gamma_1, \gamma_2 \colon G \longrightarrow \mathbb{T},$$

define $(\gamma_1, \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$ for each $x \in G$. Then $\widehat{G}$ with this product is called the dual group of G. In view of the duality between G and $\widehat{G}$, denote $\gamma(x)$ by (x, γ), for $x \in G$ and $\gamma \in \widehat{G}$. With this notation, the expression for the map

$$\begin{aligned} G \times \widehat{G} &\longrightarrow \quad \mathbb{T} \\ (x, \gamma) &\longmapsto (x, \gamma) \end{aligned}$$

makes sense and it in fact suggests that this association satisfies the bilinearity relations:

$$\begin{aligned} &(x_1 \cdot x_2, \chi) = (x_1, \chi)(x_2, \chi) \\ &(x, \chi_2 \cdot \chi_2) = (x, \chi_1)(x, \chi_2) \\ &(e, \chi) = 1, \quad (x, \widehat{e}) = 1 \end{aligned}$$

for every $x \in G$ and $\chi \in \widehat{G}$. Recall that the Guelfand spectrum $\widehat{A}$ of an abelian Banach algebra A is the set of all non-zero $\mathbb{C}$-algebra homomorphisms $A \longrightarrow \mathbb{C}$. Now, given a continuous character $\chi \in \widehat{G}$ and $f \in L^1(G)$, define

$$\widehat{f}(\chi) = \int_G f(x)\overline{(x, \chi)}\, \mathrm{d}\mu(x)$$

where μ is a Haar measure.

Theorem 7.1. The map which to every continuous character $\chi \in \widehat{G}$ associates the linear functional

$$\begin{aligned} L^1(G) &\longrightarrow \quad \mathbb{C} \\ f \quad &\longmapsto \widehat{f}(\chi) \end{aligned}$$

establishes a bijection between $\widehat{G}$ and $\widehat{L^1(G)}$.

Proof. Step 1. Given a character $\chi\colon G \longrightarrow \mathbb{T}$, the map

$$f \longmapsto \int_G f(x)\overline{(x,\chi)}\,\mathrm{d}\mu(x)$$

gives a non-zero, continuous (linear) functional on $L^1(G)$. Indeed $x \mapsto \overline{(x,\chi)}$ is in $L^\infty(G)$ and non identically zero. In addition, let $f,g \in L^1(G)$,

$$\begin{aligned}
\widehat{f*g}(\chi) &= \int_G \mathrm{d}\mu(x)\overline{(x,\chi)}\int_G f(xy)g(y^{-1})\,\mathrm{d}\mu(y)\\
&= \int_G \mathrm{d}\mu(x)\overline{(x,\chi)}\int_G f(xy^{-1})g(y)\,\mathrm{d}\mu(y)\\
&= \int_G \mathrm{d}\mu(y)g(y)\overline{(y,\chi)}\int_G \mathrm{d}\mu(x)f(xy^{-1})\overline{(xy^{-1},\chi)}\\
&= \int_G \mathrm{d}\mu(y)\overline{(y,\chi)}\int_G \mathrm{d}\mu(x)f(x)\overline{(x,\chi)}\\
&= \widehat{g}(\chi)\widehat{f}(\chi).
\end{aligned}$$

Observe that if $\chi_1,\chi_2 \in G$ define the same functional on $L^1(G)$, then they are μ-a.e. equal, since $L^\infty(G)$ is the dual of $L^1(G)$. Since open non-empty subsets of G have positive μ-measure and χ_1,χ_2 are continuous, this implies that they coincide everywhere. This shows that $\widehat{G} \hookrightarrow \widehat{L^1(G)}$.

Step 2. We have seen that every element in $\widehat{L^1(G)}$ is given by a continuous linear functional on $L^1(G)$ (Proposition 3.13). Let thus $\varphi \in \widehat{L^1(G)}$. Then there exists $\Phi \in L^\infty(G)$, $\Phi \neq 0$ such that

$$\varphi(f) = \int_G f(x)\Phi(x)\,\mathrm{d}\mu(x), \quad \forall f \in L^1(G).$$

We have to establish that this defines a continuous character. Compute:

$$\begin{aligned}
\varphi(f*g) &= \int_G \mathrm{d}\mu(x) f*g(x)\Phi(x) = \int_G \mathrm{d}\mu(x)\int_G f(xy^{-1})g(y)\,\mathrm{d}\mu(y)\Phi(x)\\
&= \int_G \mathrm{d}\mu(x)\int_G f(xy^{-1})g(y)\,\mathrm{d}\mu(y)\Phi(x)\\
&= \int_G \mathrm{d}\mu(y)g(y)\int_G \mathrm{d}\mu(x)f(xy^{-1})\Phi(x)\\
&= \int_G \mathrm{d}\mu(y)g(y)\varphi(\lambda(y)f)
\end{aligned}$$

On the other hand,

$$\varphi(f)\varphi(g) = \int_G \mathrm{d}\mu(y)g(y)\Phi(y)\varphi(f)$$

Since φ is a homomorphism, the two expressions are equal for every $f,g \in L^1(G)$, and hence for μ-a.e. $y \in G$, it holds that $\varphi(\lambda(y)f) = \Phi(y)\varphi(f)$, for every $f \in L^1(G)$. Since φ is continuous, Lemma 6.13 (ii) implies that $y \mapsto \varphi(\lambda(y)f_0)/\varphi(f_0)$ is continuous. Replacing Φ by

$\varphi(\lambda(y)f_0)/\varphi(f_0)$ implies that we may assume that Φ is continuous. Then for every $f \in L^1(G)$ (not only f_0) and almost all $y \in G$, we have

$$\varphi(\lambda(y)f) = \Phi(y)\varphi(f).$$

Again, since $y \mapsto \varphi(\lambda(y)f)$ is continuous, the equality above holds for all $y \in G$.

Step 3. Now, replacing y by $y\xi$, we write

$$\Phi(y\xi)\varphi(f) = \varphi(\lambda(y\xi)f) = \varphi(\lambda(y)(\lambda(\xi)f)) = \Phi(y)\varphi(\lambda(\xi)f) = \Phi(y)\Phi(\xi)\varphi(f)$$

and thus $\Phi(y\xi) = \Phi(y)\Phi(\xi)$ for every $y, \xi \in G$. If there were some $z \in G$ with $\Phi(z) = 0$ then it would have $\Phi(yz) = 0$ for all $y \in G$, and hence $\Phi \equiv 0$, which is a contradiction. Therefore $\Phi \colon G \to \mathbb{C}^\times$, and it is a continuous homomorphism. In addition,

$$\begin{aligned} G &\longrightarrow \mathbb{R}^*_{>0} \\ y &\longmapsto |\Phi(y)| \end{aligned}$$

has to be identically 1, since $|\Phi(y)|$ is bounded. With this, $\Phi \in \widehat{G}$ and the proof is done. □

For $f \in L^1(G)$, the function

$$\begin{aligned} \widehat{f} \colon G &\longrightarrow \mathbb{C} \\ x &\longmapsto \int_G f(x)\overline{(x,\chi)}\,\mathrm{d}\mu(x) \end{aligned}$$

is called the Fourier transform of f. We use the bijection $\widehat{G} \longrightarrow \widehat{L^1(G)}$ established in Theorem 7.1 to transport the Guelfand topology of $\widehat{L^1(G)}$ to a topology on $\widehat{G}$. In view of the definition of the Guelfand topology, a basis of open sets in $\widehat{G}$ is then given by

$$\mathcal{U}(\chi_0; f_1, \ldots, f_n; \varepsilon) = \left\{\chi \in \widehat{G} : \left|\widehat{f}_i(\chi) - \widehat{f}_i(\chi_0)\right| < \varepsilon;\ 1 \leq i \leq n\right\}$$

for $\chi_0 \in \widehat{G}$, $f_1, \ldots, f_n \in L^1(G)$ and $\varepsilon > 0$. In other words, we restrict the weak-$*$ topology of $L^\infty(G)$ to $\widehat{G} \hookrightarrow L^\infty(G)$.

Theorem 7.2.

(i) $\widehat{G}$ is a locally compact Hausdorff space.

(ii) The Fourier transform

$$\begin{aligned} L^1(G) &\longrightarrow C_0(\widehat{G}) \\ f &\longmapsto \widehat{f} \end{aligned}$$

is a norm decreasing $*$-homomorphism of Banach algebras onto a dense subalgebra $A(\widehat{G}) \subset C_0(\widehat{G})$.

(iii) $A(\widehat{G})$ is invariant under translations by $\widehat{G}$ and multiplication by $\chi \mapsto (x, \chi)$ for every $x \in G$.

(iv) For $f \in L^1(G)$ and $\chi \in \widehat{G}$,

$$(f * \chi)(x) = (x, \chi)\widehat{f}(\chi), \quad \forall x \in G.$$

Proof. First, (i) follows from Theorem 3.23. For (ii), notice that under the identification of $\widehat{G}$ with $\widehat{L^1(G)}$, the Fourier transform is really the Guelfand transform. Hence it takes values in $C_0(\widehat{G})$. Also,

$$\left|\widehat{f}(\chi)\right| = \left|\int_G f(x)\overline{(x,\chi)}\,\mathrm{d}x\right| \leq \int_G |f(x)|\,\mathrm{d}\mu(x) = \|f\|_1,$$

showing that $\|\widehat{f}\|_\infty \leq \|f\|_1$. Furthermore, $\forall \chi \in \widehat{G}\ \exists f \in L^1(G)$ with $\widehat{f}(\chi) \neq 0$, and as we observed already, for every $\chi_1 \neq \chi_2$ there exists $f \in L^1(G)$ with $\widehat{f}(\chi_1) \neq \widehat{f}(\chi_2)$. These properties could also have been deduced from the fact that $\widehat{G}$ injects into $L^\infty(G) \setminus \{0\}$.

Finally we verify that if $A(\widehat{G}) = \left\{\widehat{f} : f \in L^1(G)\right\}$ is the image of the Fourier transform, then $A(\widehat{G})$ is invariant under complex conjugation. We prove this by establishing that $f \mapsto \widehat{f}$ is a C^*-homomorphism. Let $f \in L^1(G)$,

$$\begin{aligned}\widehat{f^*}(\chi) &= \int_G \overline{f(x^{-1})(x,\chi)}\,\mathrm{d}\mu(x) = \overline{\int_G f(x^{-1})(x,\chi)\,\mathrm{d}x} \\ &= \overline{\int_G f(x)(x^{-1},\chi)\,\mathrm{d}\mu(x)} = \overline{\widehat{f}(\chi)}\end{aligned}$$

since $(x^{-1},\chi) = \chi(x^{-1}) = \chi(x)^{-1} = \overline{\chi(x)} = \overline{(x,\chi)}$. Hence $\widehat{f^*}(\chi) = \overline{\widehat{f}(\chi)}$. Therefore, by the Stone-Weierstrass theorem, $A(\widehat{G})$ is dense in $C_0(\widehat{G})$.

For (iii), an immediate computation shows that $\widehat{f}(\chi\chi_0) = \widehat{f\cdot\chi_0^{-1}}(\chi)$, which shows the first assertion. For the second one, it suffices to do a computation to verify that $\widehat{f}(\chi)(x,\chi) = \widehat{\lambda(x^{-1})f}(\chi)$. ☐

In the following proposition we can find an illustration of the dual behavior of G and $\widehat{G}$.

Proposition 7.3. If G is discrete, then $\widehat{G}$ is compact, and if G is compact then $\widehat{G}$ is discrete.

Proof.

(i) When G is discrete then $L^1(G)$ has an identity, namely δ_e. The Guelfand spectrum of $L^1(G)$ is compact, hence $\widehat{G}$ is compact.

(ii) Assume now that G is compact and let μ be the Haar probability measure on G with $\mu(G) = 1$. Then we claim that $\int_G \gamma\,\mathrm{d}\mu = \delta_{\widehat{e}}(\gamma)$. Indeed for $\gamma \neq \widehat{e}$,

$$\int_G \gamma(x)\,\mathrm{d}\mu(x) = \int_G \gamma(yx)\,\mathrm{d}\mu(x) = \int_G \gamma(y)\gamma(x)\,\mathrm{d}\mu(x);$$

this holds for all $y \in G$ only if the first term is zero. Let now $\gamma \in \widehat{G}$, consider it as in $L^1(G)$, and recall that $\widehat{\gamma} \in C_0(\widehat{G})$. Then

$$\widehat{\gamma}(\chi) = \int_G \gamma(x)\overline{(x,\chi)}\,\mathrm{d}\mu(x) = \int_G \gamma(x)\chi^{-1}(x)\,\mathrm{d}\mu(x) = \int_G (\gamma\cdot\chi^{-1})(x)\,\mathrm{d}\mu(x) = \delta_\gamma(\chi)$$

This means that $\widehat{\gamma} = \delta_\gamma$, therefore $\{\gamma\} \subset \widehat{G}$ is open for all $\gamma \in G$ meaning that G is discrete. ☐

So far we have a topology on $\widehat{G}$ making it a locally compact Hausdorff space with an abelian group structure. Our next task is to show that both structures are compatible and that $\widehat{G}$ is a locally compact Hausdorff group. This depends on an alternative description of the topology of $\widehat{G}$.

Proposition 7.4. The following statements hold

(i) The function

$$\begin{aligned} G \times \widehat{G} &\longrightarrow \mathbb{T} \\ (x, \gamma) &\longmapsto \gamma(x) \end{aligned}$$

is continuous.

(ii) Let $K \subset G$ and $C \subset \widehat{G}$ be compact subsets. Then for $\varepsilon > 0$, define

$$N(K, \varepsilon) = \left\{ \chi \in \widehat{G} : \ |(x, \chi) - 1| < \varepsilon, \, \forall x \in K \right\}$$
$$N(C, \varepsilon) = \{ x \in G : \ |(x, \chi) - 1| < \varepsilon, \, \forall \chi \in C \}$$

Then $N(K, \varepsilon) \subset \widehat{G}$ is open and $N(C, \varepsilon) \subset G$ is open.

(iii) The family of all sets $N(K, \varepsilon)$ and their translates form a basis for the given topology of $\widehat{G}$.

(iv) $\widehat{G}$ is a locally compact abelian Hausdorff group.

Proof.

(i) For $f \in L^1(G)$,

$$\begin{aligned} \widehat{\lambda(x^{-1})f}(\chi) &= \int_G f(xy)\overline{(y, \chi)} \, \mathrm{d}\mu(y) = \int_G f(y)\overline{(x^{-1}y, \chi)} \, \mathrm{d}\mu(x) \\ &= \chi(x) \int_G f(y)\overline{(y, \chi)} \, \mathrm{d}\mu(x) = \chi(x)\widehat{f}(\chi) = (x, \chi)\widehat{f}(\chi). \end{aligned}$$

If we show that for all $f \in L^1(G)$ the function

$$\begin{aligned} G \times \widehat{G} &\longrightarrow \mathbb{C} \\ (x, \chi) &\longmapsto \widehat{\lambda(x^{-1})f}(\chi) \end{aligned}$$

is continuous, then we have that the function

$$\begin{aligned} G \times \widehat{G} &\longrightarrow \mathbb{T} \\ (x, \chi) &\longmapsto \chi(x) \end{aligned}$$

is continuous on the open sets of the form

$$\left\{ (x, \chi) \in G \times \widehat{G} : \ \widehat{f}(\chi) \neq 0 \right\}, \quad f \in L^1(G),$$

which cover $G \times \widehat{G}$. This yields a proof of (i). To this end, let $(x_0, \gamma_0) \in G \times \widehat{G}$ and $\varepsilon > 0$. Define

$$V = \left\{ x \in G : \|\lambda(x^{-1})f - \lambda(x_0^{-1})f\|_1 < \varepsilon \right\},$$

an open neighbourhood of $x_0 \in G$ by Lemma 6.13 (ii), and

$$W = \left\{ \gamma \in \widehat{G} : \left| \widehat{\lambda(x_0^{-1})f}(\gamma) - \widehat{\lambda(x_0^{-1})f}(\gamma_0) \right| < \varepsilon \right\} = \mathcal{U}(\gamma_0; \lambda(x_0^{-1})f; \varepsilon)$$

is an open neighbourhood of γ_0 in $\widehat{G}$.

For any $(x, \gamma) \in V \times W$, we have

$$\begin{aligned} \left| \widehat{\lambda(x^{-1})f}(\gamma) - \widehat{\lambda(x_0^{-1})f}(\gamma_0) \right| &\leq \left| \widehat{\lambda(x^{-1})f}(\gamma) - \widehat{\lambda(x_0^{-1})f}(\gamma) \right| \\ &\quad + \left| \widehat{\lambda(x_0^{-1})f}(\gamma) - \widehat{\lambda(x_0^{-1})f}(\gamma_0) \right| \\ &\leq \| \widehat{\lambda(x^{-1})f} - \widehat{\lambda(x_0^{-1})f} \|_1 + \varepsilon < 2\varepsilon \end{aligned}$$

Since $\gamma \in W$,

$$\left| \widehat{\lambda(x_0^{-1})f}(\gamma) - \widehat{\lambda(x_0^{-1})f}(\gamma_0) \right| < \varepsilon.$$

This implies that for all $(x, \gamma) \in V \times W$, $\left| \widehat{\lambda(x^{-1})f}(\gamma) - \widehat{\lambda(x_0^{-1})f}(\gamma_0) \right| < 2\varepsilon$.

(ii) Let $K \subset G$ be a compact subset and $\varepsilon > 0$. We want to show that $N(K, \varepsilon)$ is open. Let $\chi_0 \in N(K, \varepsilon)$. For every $x_0 \in K$ there are open sets $\mathcal{U}_{x_0} \ni x_0$ and $W_{\chi_0} \ni \chi_0$ such that $|(x, \chi) - 1| < \varepsilon$ for every $x \in \mathcal{U}_{x_0}$ and $\chi \in W_{x_0}$. This follows directly from (i).

Let $y_1, ..., y_n \in K$ be such that $\bigcup_{i=1}^n U_{y_i} \supset K$ and $W = \bigcap_{i=1}^n W_{y_i}$, then $|(x, \chi) - 1| < \varepsilon$ for all $x \in K, \chi \in W \ni x_0$. This shows that $W \subset N(K, \varepsilon)$, thus making it open.

The assertion concerning $N(C, \varepsilon)$ is shown by an analogous argument.

(iii) We want to show that the $N(K, \varepsilon)$'s and their translates (by the elements of $\widehat{G}$) form a basis for the topology of $\widehat{G}$. Observe that

$$\mathcal{U}(\gamma_0; f_1, ..., f_n; \varepsilon) = \gamma_0 \, \mathcal{U}(\widehat{e}; f_1 \circ \gamma_0^{-1}, ..., f_n \circ \gamma_0^{-1}; \varepsilon).$$

Then it suffices to show that given $\varepsilon > 0$ and $f_1, ..., f_n \in L^1(G)$, there exists $r > 0$ and $K \subset G$ compact such that

$$N(K, r) \subset \mathcal{U}(\widehat{e}; f_1, ..., f_n; \varepsilon) \,.$$

For this, choose $g_1, ..., g_n \in C_{00}(G)$ with

$$\|g_i - f_i\|_1 < \varepsilon/3, \quad 1 \leq i \leq n.$$

Then $\mathcal{U}(\widehat{e}; g_1, ..., g_n; \varepsilon/3)$ is contained in $\mathcal{U}(\widehat{e}; f_1, ..., f_n; \varepsilon)$. Let now K be the union of the supports of the g_i's, which is compact by definition, and let $r < \varepsilon/3 \max_{1 \leq i \leq n} \|g_i\|_1$. For any $\gamma \in N(K, r)$,

$$\begin{aligned} |\widehat{g_i}(\gamma) - \widehat{g_i}(e)| &= \left| \int_G f_i(x) [\overline{(x, \gamma)} - 1] \, \mathrm{d}\mu(x) \right| \\ &= \int_K |g_i(x)| \, |(x, \gamma) - 1| \, \mathrm{d}\mu(x) < r \|g_i\|_1 < \varepsilon/3. \end{aligned}$$

Therefore $N(K, r) \subset \mathcal{U}(\widehat{e}; g_1, ..., g_n; \varepsilon/3)$, which proves (iii).

(iv) Let $\eta_1, \eta_2 \in \widehat{G}$,

$$|\eta_1(x)\eta_2(x) - 1| = |\eta_1(x)(\eta_2(x) - 1) + \eta_2(x) - 1| \leq |\eta_2(x) - 1| + |\eta_1(x) - 1|.$$

This implies that for every $K \subset G$ compact, and $\varepsilon > 0$,

$$N(K, \varepsilon/2) \cdot N(K, \varepsilon/2) \subset N(K, \varepsilon).$$

Now, $\forall \gamma_1, \gamma_2 \in \widehat{G}$,

$$[\gamma_1 N(K, \varepsilon/2)] \cdot [\gamma_2 N(K, \varepsilon/2)] = \gamma_1\gamma_2 N(K, \varepsilon/2)N(K, \varepsilon/2) < \gamma_1\gamma_2 N(K, \varepsilon).$$

This shows the continuity of the multiplication map $\widehat{G} \times \widehat{G} \longrightarrow \widehat{G}$. For the inverse, one verifies that

$$|\eta(x) - 1| = \left|\overline{\eta(x)} - 1\right| = \left|\eta^{-1}(x) - 1\right|,$$

so $N(K, \varepsilon)^{-1} = N(K, \varepsilon)$, meaning that

$$[\gamma N(K, \varepsilon)]^{-1} = \gamma^{-1} N(K, \varepsilon).$$

□

Remark. The fact that the sets $N(C, \varepsilon)$ together with their translates form a basis of the topology of G is highly non-trivial and will be shown in Corollary 9.11 as a consequence of the Fourier inversion theorem.

Another reformulation of the points (ii) and (ii) in the proposition is given by the following corollary.

Corollary 7.5. On $\widehat{G} = \mathrm{Hom}_{\mathrm{cont}}(G, \Pi)$ the compact-open topology and the weak-* topology induced by $\widehat{G} \hookrightarrow L^\infty(G)$ coincide.

We finish the chapter by discussing some examples.

Example 7.6.

(i) Let G_1, G_2 be locally compact, abelian Hausdorff. Then every pair $(\varphi_1, \varphi_2) \in \widehat{G_1} \times \widehat{G_2}$ gives rise to a continuous character of $G_1 \times G_2$, $(x_1, x_2) \mapsto \varphi_1(x_1)\varphi_2(x_2)$. The resulting map $\widehat{G_1} \times \widehat{G_2} \to \widehat{G_1 \times G_2}$ is a bijection.

(ii) Let $m \geq 1$ and $G = \mathbb{Z}/m\mathbb{Z}$ be the cyclic group of order m. Then the map

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} &\longrightarrow \widehat{\mathbb{Z}/m\mathbb{Z}} \\ a &\longmapsto \chi_a \end{aligned}$$

where $\chi_a(x) = e^{\frac{2\pi i a x}{m}}$ is a group isomorphism.

(iii) If F is a finite abelian group, then $\widehat{F}$ is isomorphic to F.

(iv) For every $\alpha \in \mathbb{R}$,

$$\begin{aligned} \chi_\alpha : \mathbb{R} &\longrightarrow \mathbb{T} \\ \alpha &\longmapsto e^{2\pi i \alpha x} \end{aligned}$$

is a continuous character, and the induced map $\mathbb{R} \longrightarrow \widehat{\mathbb{R}}$ is an isomorphism of locally compact abelian groups.

(v) Let $\mathbb{T} = \{\xi \in \mathbb{C}^* : |\xi| = 1\}$. There is a bijection from $\mathbb{Z}$ to $\widehat{\mathbb{T}}$ given by $m \mapsto \chi_m$, for $\chi_m(\xi) = \xi^m$.

(vi) $\mathbb{T}$ is bijective to $\widehat{\mathbb{Z}}$ via ξ giving a character $m \mapsto \xi^m$.

(vii) As a result the dual group of

$$G = \mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^l \times F,$$

where F is finite abelian, is isomorphic to

$$\widehat{G} \simeq \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^l \times F.$$

Observe that the dual thereof is

$$\widehat{\widehat{G}} \simeq \mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^l \times F,$$

and hence isomorphic to G. We will see that this is a special case of Pontryagin duality.

8 Complex measures

The material of this chapter is a summary of chapter 6 in [Ru3].

We will need to present in a more systematic way the topic of complex measures. This replaces the very ad-hoc treatment made in section 5.1.

Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$ a σ-algebra of subsets of X.

Definition 8.1. A complex measure is a set function $\mu\colon \mathcal{B} \longrightarrow \mathbb{C}$ such that whenever $E = \bigsqcup_{i\in\mathbb{N}} E_i$ with $E_i \in \mathcal{B}$ is a countable partition, then

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i),$$

and this series converges.

Remark.

(i) In contrast with the definition of positive measure, where the possible range is $[0, \infty]$, here we get $\mu(E) \in \mathbb{C}$ for every $E \in \mathcal{B}$.

(ii) If $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ is any permutation, then $E = \bigsqcup_{i=1}^{\infty} E_{\sigma(i)}$ and $\mu(E) = \sum_{i=1}^{\infty} \mu\big(E_{\sigma(i)}\big)$. Thanks to a result by Riemann, this implies that the convergence is absolute, that is $\sum_{i=1}^{\infty} |\mu(E_i)| < \infty$.

To any measure one can associate the total variation measure. A natural way to introduce this is by solving the problem of finding a positive measure λ such that $|\mu(E)| \leq \lambda(E)$ for any $E \in \mathcal{B}$. For $E = \bigsqcup_{i\in\mathbb{N}} E_i$, we would have

$$\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i) \geq \sum_{i=1}^{\infty} |\mu(E_i)| \,.$$

This suggest the following definition.

Definition 8.2. Given a complex measure μ, we define its total variation by

$$|\mu|(E) := \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| \,:\; E = \bigsqcup_{i=1}^{\infty} E_i,\; E_i \in \mathcal{B} \right\}.$$

Additionally, we set $\|\mu\| = |\mu|(X)$.

A priori this is just a set function $|\mu|\colon \mathcal{B} \longrightarrow [0,\infty]$. To the end of proving that it actually is a measure, we introduce the following lemma.

Lemma 8.3. Let $\xi_1, ..., \xi_N \in \mathbb{C}$. Then there is a subset $S \subset \{1, ..., N\}$ such that

$$\left|\sum_{i\in S} \xi_i\right| \geq \frac{1}{\pi}\sum_{i=1}^{N} |\xi| \,.$$

Proof. Let $\xi_k = |\xi_k|\, e^{i\alpha_k}$. Now, given $-\pi \leq \theta \leq \pi$, define

$$S(\theta) = \{k :\ \cos(\alpha_k - \theta) > 0,\ 1 \leq k \leq N\} \,.$$

Now we estimate

$$\begin{aligned}\left|\sum_{k\in S(\theta)} \xi_k\right| = \left|\sum_{k\in S(\theta)} \xi_k e^{-i\theta}\right| &\geq \operatorname{Re}\left(\sum_{k\in S(\theta)} \xi_k e^{-i\theta}\right) \\ &= \sum_{k=1}^{N} |\xi_k| \max\{0, \cos(\alpha_k - \theta)\} := f(\theta)\end{aligned}$$

Let θ_0 be such that $f(\theta_0) \geq f(\theta)$ for all $\theta \in [-\pi, \pi]$. We obtain

$$f(\theta_0) \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta$$

but at the same time

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \max\{0, \cos(\alpha - \theta)\}\,\mathrm{d}\theta = \frac{1}{\pi},$$

and this implies that

$$\left|\sum_{k\in S(\theta)} \xi_k\right| \geq \frac{1}{\pi}\sum_{k=1}^{N} |\xi_k| \,.$$

□

Now we are in position to prove that the total variation is actually a measure on X.

Theorem 8.4. Let $\mu\colon \mathcal{B} \longrightarrow \mathbb{C}$ be a complex measure. Then

(i) $|\mu|$ is a positive measure on $\mathcal{B}$.

(ii) $|\mu|(X) < \infty$.

Proof. The proof of (i) is very simple and is left as an exercise. We will prove (ii). Let $|\mu|$ be as in the hypotheses, called the total variation measure of μ. Assume that there is $E \in \mathcal{B}$ for which $|\mu|(E) = \infty$. Let $t = \pi(1 + |\mu(E)|)$. Since $|\mu|(E) > t$, we can find $E = \bigsqcup_{i=1}^{\infty} E_i$ and N such that $\sum_{i=1}^{N} |\mu(E_i)| > t$. By Lemma 8.3, let $S \subseteq \{1, \ldots, N\}$ be such that

$$\left|\sum_{i\in S} \mu(E_i)\right| \geq \frac{1}{\pi}\sum_{i=1}^{N} |\mu(E_i)| > \frac{t}{\pi}.$$

Let $A = \bigsqcup_{i \in S} E_i$, then $|\mu(A)| > t/\pi$. Of course, $A \subset E$. Let $B = E \setminus A$. We get

$$|\mu(B)| = |\mu(E) - \mu(A)| = |\mu(A) - \mu(E)| \geq |\mu(A)| - |\mu(E)| > \frac{t}{\pi} - |\mu(E)| = 1.$$

We conclude that if $|\mu|(E) = \infty$ then $E = A \sqcup B$ with $|\mu(A)| > 1, |\mu(B)| > 1$. Undoubtedly, $|\mu|(A) = \infty$ or $|\mu|(B) = \infty$. Assume now that $|\mu|(X) = \infty$: then writing $X = A_1 \sqcup B_1$ with $|\mu(A_1)| > 1$, $|\mu(B_1)| > 1$ and $|\mu|(B_1) = \infty$. Now split B_1 in a similar fashion. In this way we construct a sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets for which $|\mu(A_n)| > 1$. This implies that $\sum_{i=1}^{\infty} \mu(A_i)$ cannot converge. On the other hand, we should have $\mu(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ which should converge, thus reaching a contradiction. □

Notice that the space $\mathcal{M}(\mathcal{B})$ of complex measures on $\mathcal{B}$ becomes a normed vector space with the norm $\|\cdot\|$ as in definition 8.2. Also, given a complex measure μ one can always decompose it as $\mu = \mu_1 + i\mu_2$, where μ_1 and μ_2 are both complex measures with values in $\mathbb{R}$. These are called signed measures.

Definition 8.5. let $\lambda : \mathcal{B} \to \mathbb{R}$ be a signed measure, define

$$\lambda^+ = \frac{1}{2}(|\lambda| + \lambda)$$
$$\lambda^- = \frac{1}{2}(|\lambda| - \lambda);$$

then λ^+ and λ^- are positive measures with $\lambda^+(X) < \infty$, $\lambda^-(X) < \infty$, and $\lambda = \lambda^+ - \lambda^-$.

Hence a complex measure can be written as $\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$ and if $f \colon X \mapsto \mathbb{C}$ is a bounded measurable function, we define

$$\int_X f \,\mathrm{d}\mu = \left(\int_X f \,\mathrm{d}\mu_1^+ - \int_X f \,\mathrm{d}\mu_1^- \right) + i \left(\int_X f \,\mathrm{d}\mu_2^+ - \int_X f \,\mathrm{d}\mu_2^- \right).$$

This makes sense since $f \in L^1(X, \lambda)$ for any positive measure λ for which $\lambda(X) < \infty$.

If X is locally compact Hausdorff, $\mathcal{B}$ will always be the σ-algebra of Borel sets.

Definition 8.6. A complex measure $\mu \colon \mathcal{B} \longrightarrow \mathbb{C}$ will be called regular if $|\mu|$ is.

Finally, the main theorem we will discuss in this chapter is the following, and it is of the utmost relevance because of its applications.

Theorem 8.7 (*Riesz representation*). For every bounded linear functional $\Phi \colon C_0(X) \longrightarrow \mathbb{C}$ there is a unique complex regular measure μ defined on the Borel sets such that

$$\Phi(f) = \int_X f \,\mathrm{d}\mu, \quad \forall f \in C_0(X).$$

In addition, $\|\Phi\| = |\mu|(X)$.

Proof. Assume without loss of generality that $\|\Phi\| = 1$. The idea for this proof is to construct a positive linear functional $\Lambda \colon C_{00}(X) \longrightarrow \mathbb{R}$ such that

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|_\infty \quad \forall f \in C_{00}(X). \tag{8.1}$$

Let λ be the postive regular Borel measure associated to Λ by the other Riesz representation theorem. In the construction of λ, one sees that

$$\lambda(X) = \sup\left\{\Lambda(f) :\ 0 \leq f \leq 1,\ f \in C_{00}(X)\right\} \leq 1.$$

Hence we further deduce from (8.1):

$$|\Phi(f)| \leq \Lambda(|f|) = \int_X |f| \,\mathrm{d}\lambda = \|f\|_{L^1(X,\lambda)}.$$

Since $C_{00}(X)$ is dense in $L^1(X,\lambda)$, this implies that Φ extends to a bounded linear functional on $L^1(X,\lambda)$. Thus there exists a bounded Borel function $g\colon X \longrightarrow \mathbb{C}$ such that $\Phi(f) = \int_X f \cdot g \,\mathrm{d}\lambda$. Now we define $\mu = g\lambda$.

For $f \geq 0$, $f \in C_{00}(X)$, we define

$$\Lambda(f) = \sup\left\{|\Phi(h)| :\ |h| \leq f,\ h \in C_{00}(X)\right\}.$$

The main technical point is to show that if f_1, f_2 are positive, then $\Lambda(f_1+f_2) = \Lambda(f_1)+\Lambda(f_2)$. After this, one extends Λ to the real valued functions in $C_{00}(X)$ first, via $f = f^+ - f^-$, and then to the whole of $C_{00}(X)$ by linearity. □

From now on, we denote by $\mathcal{M}(X)$ the space of complex regular measures defined on the σ-algebra of Borel sets of X. Additionally, we will refer to $\mu \in \mathcal{M}(X)$ taking values in $[0,\infty)$ as bounded positive (regular) Borel measures.

Remark. Since for such a measure μ it holds that $\mu(X) < \infty$, then there is, for any $\varepsilon > 0$, a compact subset $K \subset X$ such that $\mu(K) > \mu(X) - \varepsilon$.

9 Abelian harmonic analysis

The main source for this chapter is [Ru1] 1.4, 1.5, 1.7 and 2.1. An alternative source, with a different approach is [RaVa] chapter 3. The reader of this chapter is then well prepared to read chapters 4-7 of [RaVa], or alternatively Weil's book [We].

In this chapter we will prove the basic theorems concerning Fourier analysis on locally compact abelian groups. These comprise Bochner's theorem, the Fourier inversion formula, Plancherel's theorem and Pontryagin duality.

9.1 Positive definite functions

The central result in this section will be Bochner's theorem, which characterizes continuous positive definite functions on G as Fourier transforms of positive bounded measures on $\widehat{G}$. We first turn to the notion of positive definite functions.

Definition 9.1. Let G be a group. We say that a function $\Phi \colon G \longrightarrow \mathbb{C}$ is of positive type (or positive definite) whenever each time we have $g_1, \ldots, g_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$, it holds that

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \Phi(g_j^{-1} g_i) \geq 0.$$

Example 9.2. This is the main example we will see throughout the chapter. Let $\pi \colon G \longrightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation on a Hilbert space and $v \in \mathcal{H}$, then $\Phi(g) = \langle \pi(g)v, v \rangle$ is positive definite,

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \Phi(g_j^{-1} g_i) = \left\| \sum_{i=1}^{n} c_i \pi(g_i) v \right\|^2 \geq 0.$$

Naturally, if the unitary representation $(\pi, \mathcal{H})$ is continuous in the sense that the map

$$\begin{aligned} G \times \mathcal{H} &\longrightarrow \mathcal{H} \\ (g, v) &\longmapsto \pi(g)v \end{aligned}$$

is continuous, then Φ will be continuous as well.

Let us draw a few simple conclusions from our definition.

Lemma 9.3. Let $\Phi\colon G \longrightarrow \mathbb{C}$ be positive definite.

(i) $\Phi(x^{-1}) = \overline{\Phi(x)}$ for all $x \in G$.

(ii) $|\Phi(x)| \leq \Phi(e)$ for every $x \in G$.

(iii) $|\Phi(x) - \Phi(y)|^2 \leq 2\Phi(e)\operatorname{Re}\big(\Phi(e) - \Phi(x^{-1}y)\big)$ for every $x, y \in G$.

Proof. Applying the definition to $x_1, x_2 \in G$, $c_1, c_2 \in \mathbb{C}$, we get

$$\begin{aligned}|c_1|^2\,\Phi(e)+&c_1\overline{c_2}\Phi(x_2^{-1}x_1) + c_2\overline{c_1}\Phi(x_1^{-1}x_2) + |c_2|^2\,\Phi(e) = \\ &= (|c_1|^2 + |c_2|^2)\Phi(e) + c_1\overline{c_2}\Phi(x_2^{-1}x_1) + c_2\overline{c_1}\Phi(x_1^{-1}x_2).\end{aligned}$$

With $x_1 = e$, $x_2 = x$, $c_1 = 1$ and $c_2 = c$ we get

$$(1 + |c|^2)\Phi(e) + \overline{c}\Phi(x^{-1}) + c\Phi(x) \geq 0. \tag{9.1}$$

With $c = 0$ we get $\Phi(e) \geq 0$, and with $c = 1$ we see that

$$\Phi(x^{-1}) + \Phi(x) \in \mathbb{R}.$$

With $c = i$ we see that

$$i\big(\Phi(x) - \Phi(x^{-1})\big) \in \mathbb{R},$$

and we deduce that

$$\begin{aligned}\Phi(x) + \Phi(x^{-1}) &= \overline{\Phi(x)} + \overline{\Phi(x^{-1})} \\ -\Phi(x) + \Phi(x^{-1}) &= \overline{\Phi(x)} - \overline{\Phi(x^{-1})},\end{aligned}$$

which implies $\overline{\Phi(x)} = \Phi(x^{-1})$. With this, (9.1) becomes

$$(1 + |c|^2)\Phi(e) + 2\operatorname{Re}(c\Phi(x)) \geq 0 \quad \forall c \in \mathbb{C}.$$

Choosing c such that $c\Phi(x) = -|\Phi(x)|$ we obtain $2\Phi(e) - 2\,|\Phi(x)| \geq 0$. This shows (i) and (ii).

Now, for $x_1, x_2, x_3 \in G$ and $c_1, c_2, c_3 \in \mathbb{C}$, taking (i) into account, the expression reduces to

$$\begin{aligned}\Big(|c_1|^2 + |c_2|^2 + |c_3|^2\Big)&\Phi(e) + 2\operatorname{Re}\big(\overline{c_1}c_2\Phi(x_1^{-2}x_2)\big) \\ &+ 2\operatorname{Re}\big(\overline{c_1}c_3\Phi(x_1^{-1}x_3)\big) + 2\operatorname{Re}\big(\overline{c_2}c_3\Phi(x_2^{-1}x_3)\big) \geq 0.\end{aligned}$$

Taking $x_1 = e$, $x_2 = x$, $x_3 = y$, $c_1 = 1$, $c_3 = -c_2$, we get

$$\Big(1 + 2\,|c_2|^2\Big)\,\Phi(e) + 2\operatorname{Re}\left(c_2\Phi(x)\right) - 2\operatorname{Re}\left(c_2\Phi(y)\right) - 2\,|c_2|^2\operatorname{Re}\Phi(x^{-1}y) \geq 0.$$

That is

$$2\,|c_2|^2\left[\Phi(e) - \operatorname{Re}\Phi(x^{-1}y)\right] + 2\operatorname{Re}\left[c_2(\Phi(x) - \Phi(y))\right] + \Phi(e) \geq 0.$$

Set $x_2(\Phi(x) - \Phi(y)) = \lambda\,|\Phi(x) - \Phi(y)|$ with $\lambda \in \mathbb{R}$. Then we get

$$2\lambda^2\big(\Phi(e) - \operatorname{Re}\Phi(x^{-1}y)\big) + 2\lambda\,|\Phi(x) - \Phi(y)| + \Phi(e) \geq 0, \quad \forall\lambda \in \mathbb{R}.$$

Writing that the discriminant has to be smaller or equal to 0 proves (iii). □

Corollary 9.4. Assume G is locally compact Hausdorff. If $\Phi\colon G \longrightarrow \mathbb{C}$ is positive definite and continuous at $e \in G$, then it is left and right uniformly continuous on G.

From now on we may assume that G is locally compact abelian. Then as shown in Proposition 7.4, its dual $\widehat{G}$ is locally compact abelian as well. The next example of positive definite continuous function is central.

Example 9.5. Let μ be a positive bounded measure on $\widehat{G}$. Then

$$\Phi(x) = \int_{\widehat{G}} (x, \chi)\, \mathrm{d}\mu(\chi)$$

is continuous and positive definite. Observe that since $|(x, \chi)| = 1$ when $x \in G$ and $\chi \in \widehat{G}$, the function $\chi \mapsto (x, \chi)$ is in $L^1(\widehat{G}, \mu)$. Next, if $x_1, ..., x_n \in G$ and $c_1, ..., c_n \in \mathbb{C}$, then we have

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \Phi(x_j^{-1} x_i) = \int_{\widehat{G}} \sum_{i,j=1}^{n} c_i \overline{c_j} \underbrace{(x_j^{-1} x_i, \chi)}_{=(x_i,\chi)\overline{(x_j,\chi)}} \mathrm{d}\mu(x) = \int_{\widehat{G}} \left| \sum_{i=1}^{n} c_i (x_i, \chi) \right|^2 \mathrm{d}\mu(x) \geq 0$$

which shows that Φ is positive definite. Next we show continuity at e. For that, let $\varepsilon > 0$ and $C \subset \widehat{G}$ be a compact subset with $\mu(C) > \mu(\widehat{G}) - \varepsilon$. Then

$$\begin{aligned} |\Phi(x) - \Phi(e)| &= \left| \int_{\widehat{G}} ((x, \chi) - 1)\, \mathrm{d}\mu(\chi) \right| \leq \int_{\widehat{G}} |(x, \chi) - 1|\, \mathrm{d}\mu(\chi) \\ &= \int_C + \int_{\widehat{G} \setminus C} \leq \int_C |(x, \chi) - 1|\, \mathrm{d}\mu(\chi) + 2\varepsilon \end{aligned}$$

but by Proposition 7.4,

$$N\Big(C, \frac{\varepsilon}{\mu(C)}\Big) = \Big\{ x \in G : \ |(x, \chi) - 1| < \frac{\varepsilon}{\mu(C)} \quad \forall \chi \in \widehat{G} \Big\}$$

is an open neighbourhood of e in G. Thus, for every $x \in N\Big(C, \frac{\varepsilon}{\mu(C)}\Big)$,

$$|\Phi(x) - \Phi(e)| \leq \int_C \frac{\varepsilon}{\mu(C)}\, \mathrm{d}\mu(\chi) + 2\varepsilon = 3\varepsilon.$$

By the previous lemma, Φ is continuous everywhere.

Now we have a chance to prove Bochner's theorem, which essentially tells us that every continuous, positive definite function on a locally compact abelian group is the Fourier transform of a positive bounded measure on the dual group.

Theorem 9.6 (*Bochner*). Let G be a locally compact abelian group and $\Phi\colon G \longrightarrow \mathbb{C}$ continuous positive definite. Then there is a unique positive bounded measure μ on $\widehat{G}$ such that

$$\Phi(x) = \int_{\widehat{G}} (x, \chi)\, \mathrm{d}\mu(\chi) \quad \forall x \in G.$$

Proof. We may assume that $\Phi(e) > 0$, since otherwise by Lemma 9.3 (ii), $\Phi = 0$, and it suffices to take the zero measure to prove the theorem. Then, rescaling Φ by multiplying it by a positive constant we may assume $\Phi(e) = 1$. Now, let

$$T_\Phi(f) = \int_G f(x)\Phi(x)\,\mathrm{d}\lambda(x), \quad f \in L^1(G).$$

where λ is a fixed Haar measure on G. This defines a continuous linear functional on $L^1(G)$ with $\|T_\Phi\| = \|\Phi\|_\infty = \Phi(e) = 1$.

Define on $L^1(G)$ a hermitian form

$$[f, g] = T_\Phi(f * g^*),$$

where $g^*(x) = \overline{g(x^{-1})}$, the involution on the abelian Banach algebra $L^1(G)$. We begin with a couple of claims.

Claim (*First*)

We claim that $[f, f] \geq 0$ for every $f \in L^1(G)$. For this, we first need to obtain a different expression for $[f, f]$.

$$\begin{aligned}[f, f] = T_\Phi(f * f^*) &= \int_G \mathrm{d}\lambda(x)\Phi(x) \int_G f(xy)\overline{f(y)}\,\mathrm{d}\lambda(y) \\ &= \int_G \mathrm{d}\lambda(y)\overline{f(y)} \int_G \mathrm{d}\lambda(x)\Phi(x)f(xy) = \int_G \mathrm{d}\lambda(x) \int_G \mathrm{d}\lambda(y)f(x)\overline{f(y)}\Phi(y^{-1}x).\end{aligned}$$

First, if $f \in C_{00}(G)$, then using uniform continuity of $(x, y) \mapsto f(x)\overline{f(y)}\Phi(y^{-1}x)$ on $K \times K$, where $K = \operatorname{supp} f$, we can partition $K = \bigsqcup_{i=1}^n E_i$, the E_i being bounded sets such that the Riemann sum

$$\sum_{i,j=1}^n f(x_i)\overline{f(x_j)}\Phi(x_j^{-1}x_i)\lambda(E_i)\lambda(E_j), \quad x_i \in E_i$$

approximates the integral and hence

$$0 \leq \int\int \mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y)f(x)\overline{f(y)}\Phi(y^{-1}x).$$

By density of $C_{00}(G)$ in $L^1(G)$, and since $f \mapsto T_\Phi(f * f^*)$, we conclude the first claim.

Since $[\cdot, \cdot]$ is a hermitian form on $L^1(G)$ with $[f, f] \geq 0$ for all functions $f \in L^1(G)$, we conclude

$$|[f, g]|^2 \leq [f, f][g, g] \quad \forall f, g \in L^1(G).$$

Claim (*Second*)

Next, we claim that

$$|T_\Phi(f)|^2 \leq [f, f] = T_\Phi(f * f^*), \quad \forall f \in L^1(G).$$

We compute

$$[f, g] = \int_G \mathrm{d}\lambda(x) \int_G \mathrm{d}\lambda(y)f(x)\overline{g(y)}\Phi(y^{-1}x) = \int_G \mathrm{d}\lambda(x)f(x) \int_G \mathrm{d}\lambda(y)\overline{g(y)}\Phi(y^{-1}x).$$

Take now $w = \chi_V/\lambda(V)$, where $V = V^{-1} \ni e$, open with $\overline{V}$ a compact set. Then

$$[f,g] - T_\Phi(f) = \int_G \mathrm{d}\lambda(x) f(x) \frac{1}{\lambda(V)} \int_V \mathrm{d}\lambda(y) \big(\Phi(y^{-1}x) - \Phi(x)\big)\,,$$

and

$$[g,g] - 1 = \frac{1}{\lambda(V)^2} \int_V \int_V [\Phi(y^{-1}x) - 1]\, \mathrm{d}\lambda(x)\, \mathrm{d}\lambda(y),$$

and since $\Phi(e) = 1$ and Φ is uniformly continuous (by Lemma 9.3), both expressions can be made arbitrarily small by taking V small enough. This proves the claim.

Set now $h = f * f^*$, for $f \in L^1(G)$. Then $h^* = h$, and set

$$h^n = h^{n-1} * h, \quad n = 2, 3, 4.$$

From the claims we obtain

$$|T_\Phi(h)|^2 \le T_\Phi(h * h) = T_\Phi(h^2)$$

and

$$\left|T_\Phi(h^2)\right|^2 \le T_\Phi(h^2 * h^2) = T_\Phi(h^4)$$

on account that $h = h^*$. Combining the previous,

$$|T_\Phi(f)|^2 \le |T_\Phi(h)| \le \left|T_\Phi(h^2)\right|^{1/2} \le \left|T_\Phi(h^4)\right|^{1/4}$$

and

$$|T_\Phi(f)|^2 \le \left|T_\Phi(h^{2^n})\right|^{1/2^n} \le \|h^{2^n}\|_1^{1/2^n}$$

and this tends to $\|h\|_{\mathrm{Sp}}$ as $n \to \infty$ by Corollary 2.6. Now, by Theorem 3.23 and Theorem 7.1, $\|\widehat{h}\|_\infty = \|h\|_{\mathrm{Sp}}$, and therefore

$$h = f * f^*, \quad \widehat{h} = \left|\widehat{f}\right|^2$$

$$\|\widehat{h}\|_\infty = \|\widehat{f}\|_\infty^2, \quad |T_\Phi(f)| \le \|\widehat{f}\|_\infty.$$

In particular, if $\widehat{f} = 0$, then $T_\Phi(f) = 0$. Thus, since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$, T_Φ extends to a bounded linear functional on $C_0(\widehat{G})$ and hence there is a complex measure μ on $\widehat{G}$ such that

$$T_\Phi(f) = \int_{\widehat{G}} \widehat{f}(\gamma^{-1})\, \mathrm{d}\mu(\gamma) = \int_G f(x) \int_{\widehat{G}} (x,\gamma)\, \mathrm{d}\mu(\gamma).$$

Therefore $\Phi(x)$ is given by integration over $\widehat{G}$ of (x,γ) against μ for almost all $x \in G$, and hence for all x both sides of the equality are continuous.

Finally, we show that μ is positive. We have

$$0 \le T_\Phi(f * f^*) = \int_{\widehat{G}} \left|\widehat{f}(\gamma^{-1})\right|^2 \mathrm{d}\mu(\gamma).$$

Now the image of the continuous map

$$\begin{aligned} C_0(\widehat{G}) &\longrightarrow C_0(\widehat{G}) \\ \varphi &\longmapsto |\varphi|^2 \end{aligned}$$

is precisely the set $\left\{\varphi \in C_0(\widehat{G}) :\ \varphi \geq 0\right\}$, and since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$, we deduce

$$\int_G \varphi \,\mathrm{d}\mu \geq 0, \quad \forall \varphi \in C_0(\widehat{G}),\ \phi \geq 0.$$

□

Let G be a locally compact abelian group, and denote

$$P(G) = \{f\colon G \longrightarrow \mathbb{C} :\ f \text{ is continuous and of positive type}\}.$$

This is a convex cone. Now, if $B(G)$ is the $\mathbb{C}$-vector space generated by $P(G)$, then $B(G)$ is a sub-vector space of the space of all bounded continuous functions on G. For a complex measure μ on $\widehat{G}$, let

$$\tilde{\mu}(x) = \int_{\widehat{G}} (x, \chi) \,\mathrm{d}\mu(\chi).$$

For the moment, we can draw the following conclusion from Bochner's theorem.

Corollary 9.7. The map $\mu \mapsto \tilde{\mu}$ is a bijection $\mathcal{M}(\widehat{G}) \longrightarrow B(G)$.

Proof. *Step 1.* Let $f \in L^1(G)$. Since $\tilde{\mu}$ is continuous and bounded,

$$\begin{aligned} \int_G f(x)\tilde{\mu}(x) \,\mathrm{d}\lambda(x) &= \int_G \mathrm{d}\lambda(x) f(x) \int_{\widehat{G}} (x, \chi) \,\mathrm{d}\mu(\chi) \\ &= \int_{\widehat{G}} \mathrm{d}\mu(\chi) \int_G f(x)(x, \chi) \,\mathrm{d}\lambda(x) = \int_{\widehat{G}} \mathrm{d}\mu(\chi) \widehat{f}(\chi^{-1}). \end{aligned}$$

If $\tilde{\mu} = 0$, then $\int_{\widehat{G}} \widehat{f}(\chi^{-1}) \,\mathrm{d}\mu(\chi) = 0$, but $A(\widehat{G}) = \left\{\widehat{f} :\ f \in L^1(G)\right\}$ is dense in $C_0(\widehat{G})$. This implies $\mu = 0$.

Step 2. The map takes its values in $B(G)$,

$$\mu = \mu_1 + i\mu_2 = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-).$$

Hence the same decomposition takes place with $\tilde{\mu}$ and we have seen that if α is a positive bounded measure then $\tilde{\alpha} \in P(G)$. Hence $\tilde{\mu} \in B(G)$. Finally the subjectivity follows from Bochner's theorem. □

9.2 The inversion theorem

Theorem 9.8. Given a Haar measure λ on G there exists a unique Haar measure ω on $\widehat{G}$ such that

(i) if $f \in L^1(G) \cap B(G)$ then $\widehat{f} \in L^1(\widehat{G})$.

(ii) for every $f \in L^1(G) \cap B(G)$ we have

$$f(x) = \int_{\widehat{G}} \widehat{f}(\chi)(x, \chi) \,\mathrm{d}\omega(\chi), \quad \forall x \in G.$$

Remark. From the hypotheses $f \in L^1(G) \cap B(G)$ we have, from Corollary 9.7 that there is $\mu_f \in \mathcal{M}(\widehat{G})$ such that

$$f(x) = \int_{\widehat{G}} (x, \chi)\, \mathrm{d}\mu_f(\chi).$$

We also know that $\widehat{f}$ is well defined and in $C_0(\widehat{G})$. The inversion theorem amounts to proving that $\mu_f = \widehat{f}\, \mathrm{d}\omega$. We have to show that $\mu_f / \widehat{f}$ defines a Haar measure on $\widehat{G}$.

Lemma 9.9. For every $f, g \in L^1(G) \cap B(G)$, we have

$$\widehat{f}\, \mathrm{d}\mu_g = \widehat{g}\, \mathrm{d}\mu_f.$$

Proof. Let $h \in L^1(G)$. We compute

$$h * f(e) = \int_G \mathrm{d}\lambda(x) h(x) f(x^{-1}) = \int_G \mathrm{d}\lambda(x) h(x) \int_{\widehat{G}} (x^{-1}, \chi)\, \mathrm{d}\mu_f(\chi) = \int_{\widehat{G}} \mathrm{d}\mu_f(\chi) \widehat{h}(\chi).$$

Now replacing h by $h * g$ we obtain

$$\int_{\widehat{G}} \mathrm{d}\mu_f \widehat{h}(\chi) \widehat{g}(\chi) = (h * g) * f(e) = (h * f) * g(e) = \int_{\widehat{G}} \mathrm{d}\mu_g(\chi) \widehat{h}(\chi) \widehat{f}(\chi).$$

By arbitrariness of h and since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$ the proof is complete. □

This lemma says that formally, $\mathrm{d}\mu_f / \widehat{f}$ is independent of g. The problem with the strategy mentioned above for proving the inversion formula is the fact that $\widehat{f}$ can have zeroes. The following lemma gives a way to fix this issue, and for its proof we will need the following observation.

Remark. Let $f \in L^2(G)$. Then $f * f^*$ is well defined and it belongs in $P(G)$. Hence if $f \in C_{00}(G)$ then $f * f^*$ lies in $L^1(G) \cap P(G)$. Indeed

$$f * f^*(x) = \int_G f(xy) \overline{f(y)}\, \mathrm{d}\lambda(y) = \langle \lambda(x^{-1}) f, f \rangle,$$

where $\lambda \colon G \longrightarrow \mathcal{U}(L^2(G))$ is a continuous unitary representation and hence $x \mapsto \langle \lambda(x^{-1}) f, f \rangle$ is in $P(G)$.

Lemma 9.10. Given any compact set $K \subset \widehat{G}$, there is $g \in C_{00}(G) \cap P(G)$ such that $\widehat{g} > 0$ on K.

Proof. We know that for any $\gamma \in K$ there is a function $u_\gamma \in C_{00}(G)$ with $\widehat{u_\gamma}(\gamma) \neq 0$. Then there is an open set $V_\gamma \ni \gamma$ with $\widehat{u}_\gamma(\eta) \neq 0$ for every $\eta \in V_\gamma$. Now, $K \subset \bigcup_{\gamma \in K} V_\gamma$, and since it is compact, we can reduce these to finitely many γ's, namely $\gamma_1, ..., \gamma_n \in K$ with $K \subset \bigcup_{i=1}^n V_{\gamma_i}$. Let then

$$g := \sum_{i=1}^n u_{\gamma_i} * u_{\gamma_i}^* \in C_{00}(G) \cap P(G).$$

It follows that

$$\widehat{g}(\eta) = \sum_{i=1}^n |u_{\gamma_i}(\eta)|^2 > 0, \quad \forall \eta \in K.$$

□

We now come to the proof of Theorem 9.8.

Proof of Theorem 9.8. Let $\psi \in C_{00}(\widehat{G})$ and pick $K \supset \operatorname{supp}\psi$ compact and $g \in C_{00}(G) \cap P(G)$ with $\widehat{g} > 0$ on K. Define

$$T_{K,g}(\psi) := \int_K \frac{\psi(\chi)}{\widehat{g}(\chi)}\, \mathrm{d}\mu_g(\chi).$$

This does not depend on K or g. Indeed, if $K' \supset \operatorname{supp}\psi$ is another compact set and $g' \in C_{00}(G) \cap P(G)$ are such that $\widehat{g'} > 0$ on K', then

$$T_{K,g}(\psi) = \int_K \frac{\psi(\chi)}{\widehat{g}(\chi)}\, \mathrm{d}\mu_g(\chi) = \int_{K\cap K'} \frac{\psi(\chi)}{\widehat{g}(\chi)} \frac{\widehat{g'}(\chi)}{\widehat{g'}(\chi)}\, \mathrm{d}\mu_g(\chi),$$

and since $\widehat{g'}\mu_g = \widehat{g}\mu_{g'}$ this equals

$$\int_{K\cap K'} \frac{\psi(\chi)}{\widehat{g'}(\chi)}\, \mathrm{d}\mu_{g'}(\chi) = \int_{K'} \frac{\psi(\chi)}{\widehat{g'}(\chi)}\, \mathrm{d}\mu_{g'}(\chi) = T_{K',g'}(\psi).$$

Now we write

$$T(\psi) = \int_K \frac{\psi(\chi)}{\widehat{g}(\chi)}\, \mathrm{d}\mu_g(\chi),$$

where K and g are as above. We claim that $T : C_{00}(\widehat{G}) \longrightarrow \mathbb{C}$ is a positive linear and $\widehat{G}$-invariant functional.

Linearity. Letting $\psi_1, \psi_2 \in C_{00}(\widehat{G})$ and $K \supset \operatorname{supp}\psi_1 \cup \operatorname{supp}\psi_2$ compact, $g \in C_{00}(G) \cap P(G)$ with $\widehat{g} > 0$ on K,

$$\begin{aligned} T(\psi_1 + \psi_2) &= \int_K \frac{\psi_1(\chi) + \psi_2(\chi)}{\widehat{g}(\chi)}\, \mathrm{d}\mu_g(\chi) \\ &= \int_K \frac{\psi_1(\chi)}{\widehat{g}(\chi)}\, \mathrm{d}\mu_g(\chi) + \int_K \frac{\psi_2(\chi)}{\widehat{g}(\chi)}\, \mathrm{d}\mu_g(\chi) = T(\psi_1) + T(\psi_2). \end{aligned}$$

Positivity. Since $\widehat{g} > 0$ on K and $g \in P(G)$, μ_g is a positive bounded measure. Therefore $T(\psi) \geq 0$ whenever $\psi \geq 0$.

Invariance. Let $\psi \in C_{00}(\widehat{G})$ and $\gamma_0 \in \widehat{G}$. Choose $K \supset \operatorname{supp}\psi$ compact and $g \in C_{00}(G) \cap P(G)$ with $\widehat{g} > 0$ on $K \cup \gamma_0 K$. Let $f(x) = \overline{(x, \gamma_0)}g(x)$. Then

$$f(x) = \overline{(x, \gamma_0)} \int_{\widehat{G}} (x, \chi)\, \mathrm{d}\mu_g(\chi) = \int_{\widehat{G}} (x, \gamma_0^{-1}\chi)\, \mathrm{d}\mu_g(\chi).$$

Hence $f(x) = \int_{\widehat{G}} (x, \chi)\, \mathrm{d}\mu_g(\chi) = \tilde{\mu}_f(x)$ with

$$\int_{\widehat{G}} \varphi(\chi)\, \mathrm{d}\mu_f(\chi) = \int_{\widehat{G}} \varphi(\gamma_0^{-1}\chi)\, \mathrm{d}\mu_g(\chi), \quad \forall \varphi \in C_0(\widehat{G}).$$

We write

$$T(\lambda(\gamma_0)\psi) = \int \frac{\psi(\gamma_0^{-1}\gamma)}{\widehat{g}(\gamma)}\, \mathrm{d}\mu_g(\gamma) = \int \frac{\psi(\gamma)}{\widehat{g}(\gamma_0\gamma)}\, \mathrm{d}\mu_f(\gamma) = \int \frac{\psi(\gamma)}{\widehat{f}(\gamma)}\, \mathrm{d}\mu_f(\gamma) = T(\psi)$$

since

$$\widehat{g}(\gamma_0\gamma) = \int_{\widehat{G}} \overline{(x, \gamma_0)(x, \gamma)}g(x)\, \mathrm{d}\lambda(x) = \int_G f(x)\overline{(x, \gamma)}\, \mathrm{d}\lambda(x) = \widehat{f}(\gamma).$$

Let thus ω be the corresponding Haar measure representing T. Let $\psi \in C_{00}(\widehat{G})$ and $f \in L^1(G) \cap B(G)$. Pick $g \in C_{00}(G) \cap P(G)$ such that $\widehat{g} > 0$ on $K = \operatorname{supp} \psi$. Then

$$\int \psi \, \mathrm{d}\mu_f = \int \frac{\psi}{\widehat{g}} \widehat{g} \, \mathrm{d}\mu_f = \int \frac{\psi}{\widehat{g}} \widehat{f} \, \mathrm{d}\mu_g = T(\psi \widehat{f}) = \int \psi \widehat{f} \, \mathrm{d}\omega.$$

We conclude that $\mu_f = \widehat{f} \, \mathrm{d}\omega$, which proves the inversion formula. □

We turn to an important consequence of the inversion theorem. Recall from Chapter 7 that for $C \subset \widehat{G}$ compact and $r > 0$ we have defined

$$N(C, r) = \{x \in G : \ |(x, \gamma) - 1| < r \quad \forall \gamma \in C\},$$

and have shown that it is an open set. In fact, it is an open neighborhood of $e \in G$. We have the following result.

Corollary 9.11. The set

$$\left\{ gN(C, r) : \ g \in G, C \subset \widehat{G} \text{ compact}, r > 0 \right\}$$

is a basis of the topology of G.

Proof. We will show that every neighbourhood of the identity contains one of these sets, from which the corollary follows. Let $V \ni e$ be an open neighborhood of e and $W \ni e$ a compact neighborhood of the identity such that $W \cdot W^{-1} \subset V$. Let $f = \chi_W / \sqrt{\lambda(W)}$ and $g = f * f^*$. The strategy is to find $C \subset \widehat{G}$ compact such that if $x \in N(C, 1/3)$, then $g(x) > 0$. This implies $x \in \operatorname{supp} g \subset W \cdot W^{-1} \subset V$, and thus $N(C, 1/3) \subset V$.

Now, $g \in C_{00}(G)$, since it is the convolution of two compactly supported functions, and additionally $g \in P(G)$: we can apply the inversion theorem,

$$g(x) = \int_{\widehat{G}} \widehat{g}(\gamma)(x, \gamma) \, \mathrm{d}\omega(\gamma), \tag{9.2}$$

with ω the Haar measure on $\widehat{G}$ for which the inversion formula holds. For $x = e$, we get

$$\begin{aligned} g(e) = f * f^*(e) &= \int_G |f(y)|^2 \, \mathrm{d}y \\ &= \int_W \frac{1}{\lambda(W)} \, \mathrm{d}\lambda(y) = 1 \end{aligned}$$

Now, $\widehat{g}(\gamma) = \left| \widehat{f}(\gamma) \right|^2 \geq 0$, and together with (9.2), we have $1 = \int_{\widehat{G}} \widehat{g}(\gamma) \, \mathrm{d}\omega(\gamma)$. Let $C \subset \widehat{G}$ be compact with

$$\int_C \widehat{g}(\gamma) \, \mathrm{d}\omega(\gamma) > \frac{2}{3}.$$

Let $x \in N(C, 1/3)$, and write (9.2),

$$\begin{aligned} g(x) &= \int_C \widehat{g}(\gamma)(x, \gamma) \, \mathrm{d}\omega(\gamma) + \int_{\widehat{G} \setminus C} \widehat{g}(\gamma)(x, \gamma) \, \mathrm{d}\omega(\gamma) \\ &= \int_C \widehat{g}(\gamma) \operatorname{Re}(x, \gamma) \, \mathrm{d}\omega(\gamma) + \int_{\widehat{G} \setminus C} \widehat{g}(\gamma) \operatorname{Re}(x, \gamma) \, \mathrm{d}\omega(\gamma) \end{aligned}$$

Since the second term in the right-hand side is smaller than $1/3$ in absolute value, and since $|1-(x,\gamma)| < 1/3$ for every $\gamma \in C$ and $x \in N(C,1/3)$, then $\mathrm{Re}\,(x,\gamma) > 2/3$, and we can estimate the first term in the right-hand side,

$$\int_C \widehat{g}(\gamma)\,\mathrm{Re}(x,\gamma)\,\mathrm{d}\omega(\gamma) > \frac{2}{3}\int_C \widehat{g}(\gamma)\,\mathrm{d}\omega(\gamma) > \frac{4}{9}.$$

We end up with $g(x) > 4/9 - 1/3 = 1/9 > 0$. □

The next corollary, is one of the key results that will later allow us to prove Pontryagin duality.

Corollary 9.12. The set $\widehat{G}$ separates points in G. This means, that for all $x_1 \neq x_2$ in G there exists $\chi \in \widehat{G}$ with $\chi(x_1) \neq \chi(x_2)$.

Proof. We have $x_1^{-1}x_2 \neq e$, so we may pick an open neighborhood V of e with $x_1^{-1}x_2 \notin V$ since G is Hausdorff. By Corollary 9.11 we may find a non-empty, compact set $C \subset \widehat{G}$ with $N(C,1/3) \subset V$. Hence $x_1^{-1}x_2 \notin N(C,1/3)$, and therefore there exists $\chi \in C$ such that $\left|(x_1^{-1}x_2,\chi) - 1\right| \geq 1/3$, implying that $\chi(x_1) \neq \chi(x_2)$. □

Example 9.13.

(i) Let $G = \mathbb{R}$, $\chi(t) = \exp(2\pi i t)$, and identify $\mathbb{R}$ and $\widehat{\mathbb{R}}$ via $a \mapsto \chi_a$, being $\chi_a(t) = \chi(at)$. With this, if $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$, then

$$\widehat{f}(a) = \int_{\mathbb{R}} f(t)e^{-2\pi i a t}\,\mathrm{d}\mathcal{L}(t), \quad f(t) = \int_{\mathbb{R}} \widehat{f}(a)e^{2\pi i a t}\,\mathrm{d}\mathcal{L}(a),$$

for $f \in L^1(\mathbb{R}) \cap P(\mathbb{R})$.

(ii) Take $G = \mathbb{Q}_p$. Fix a continuous character $\chi\colon \mathbb{Q}_p \longrightarrow \mathbb{T}$ with $\ker\chi = \mathbb{Z}_p$. This is possible since the discrete group $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to a subgroup of $\mathbb{T}$, namely

$$\left\{\xi \in \mathbb{T} :\ \exists n \geq 1,\ \xi^{p^n} = 1\right\}$$

Identify $\widehat{\mathbb{Q}_p}$ with $\mathbb{Q}_p$ via $a \mapsto \chi_a$ where $\chi_a(t) = \chi(at)$.

Let λ be the Haar measure on $\mathbb{Q}_p$ with $\lambda(\mathbb{Z}_p) = 1$. Then λ is also the right normalization for the dual Haar measure. (Hint: $\widehat{\chi_{\mathbb{Z}_p}} = \chi_{\mathbb{Z}_p}$).

(iii) If G is compact, we have seen that $\widehat{G}$ is discrete. Let λ be the Haar measure on G with $\lambda(G) = 1$. We have see that in this case

$$\widehat{\mathbf{1}_G}(\chi) = \begin{cases} 1, & \chi = \widehat{e} \\ 0, & \chi \neq \widehat{e}. \end{cases}$$

Hence $\mathbf{1}_G = \sum_{\gamma \in \widehat{G}} \delta_{\widehat{e}}(\gamma)$, which implies that the dual measure ω on $\widehat{G}$ is the counting measure.

9.3 The Plancherel Theorem

In this section we will prove Plancherel's Theorem and draw some consequences from it, among which there is a characterization of $A(\widehat{G})$ in terms of $L^2(\widehat{G})$. We fix Haar measures λ on G and ω on $\widehat{G}$ such that the inversion theorem holds.

Theorem 9.14 (*Plancherel*). The Fourier transform $f \mapsto \widehat{f}$ defined on $L^1(G) \cap L^2(G)$ extends to an isometric isomorphism $L^2(G) \longrightarrow L^2(\widehat{G})$.

Proof. Let $f \in L^1(G) \cap L^2(G)$, then $g = f * f^* \in L^1(G) \cap P(G)$. The inversion theorem holds and in particular

$$\int_G |f(x)|^2 \,\mathrm{d}\lambda(x) = g(e) = \int_{\widehat{G}} \widehat{g}(\gamma) \,\mathrm{d}\omega(\gamma) = \int_{\widehat{G}} \left|\widehat{f}(\gamma)\right|^2 \mathrm{d}\omega(\gamma).$$

Hence $\|f\|_2^2 = \|\widehat{f}\|_2^2$ for every $f \in L^1(G) \cap L^2(G)$. It remains to show that the subspace $\mathcal{L} = \left\{\widehat{f} : f \in L^1(G) \cap L^2(G)\right\}$ is dense in $L^2(\widehat{G})$.

Let $\psi \in L^2(\widehat{G})$ and assume

$$\int_{\widehat{G}} \widehat{f}(\gamma)\overline{\psi(\gamma)} \,\mathrm{d}\omega(\gamma) = 0, \quad \forall f \in L^1(G) \cap L^2(G).$$

Since $L^1(G) \cap L^2(G)$ is translation invariant, the function $\gamma \mapsto (x, \gamma)\widehat{f}(\gamma)$ is also in $\mathcal{L}$ for each $x \in G$. Therefore

$$\int_{\widehat{G}} \widehat{f}(\gamma)\overline{\psi(\gamma)}(x, \gamma) \,\mathrm{d}\omega(\gamma) = 0 \quad \forall f \in L^1(G) \cap L^2(G), \forall x \in G.$$

Now observe that $\mu := \widehat{f}\psi\omega \in \mathcal{M}(\widehat{G})$ since both functions belong to L^2 and thus their product is in L^1. Then $\tilde{\mu} = 0$, where

$$\tilde{\mu}(x) = \int_{\widehat{G}} (x, \gamma) \,\mathrm{d}\gamma.$$

In Corollary 9.7 we showed that the map $\mu \mapsto \tilde{\mu}$ is an isomorphism, so by the injectivity this implies that $\mu = 0$ and thus for all $f \in L^1(G) \cap L^2(G)$ it holds that $\widehat{f} \cdot \psi = 0$ almost everywhere. Now, by Lemma 9.10, given any $V \subset G$ open with $\overline{V}$ compact, there exists $g \in C_{00}(G)$ with $\widehat{g} > 0$ on $\overline{V}$. Therefore $\psi = 0$ a.e. and the theorem is proved. □

We will again denote by $f \mapsto \widehat{f}$ the isomorphism of Hilbert spaces $L^2(G) \longrightarrow L^2(\widehat{G})$. Notice that care must be taken since if $f \in L^2(G) \setminus L^1(G)$, then $\widehat{f}$ is not given by the familiar integral formula.

Corollary 9.15 (*Parseval's formula*). For every $f, g \in L^2(G)$ we have

$$\int_G f(x)\overline{g(x)} \,\mathrm{d}\lambda(x) = \int_{\widehat{G}} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)} \,\mathrm{d}\omega(\gamma).$$

The proof is simple: a linear isometry preserves the norm, and hence also the scalar product.

Theorem 9.16. The following holds true

$$A(\widehat{G}) = \left\{F_1 * F_2 : \; F_1, F_2 \in L^2(\widehat{G})\right\}.$$

Proof. Recall from Theorem 6.14 that for any $F_1, F_2 \in L^2(\widehat{G})$, $F_1 * F_2$ is well defined and lies in $C_0(\widehat{G})$. Let $f, g \in L^2(G)$, and $u(x) = g(x)\overline{(x, \gamma_0)}$. We compute

$$\begin{aligned}\int_G f(x)g(x)\overline{(x,\gamma_0)}\,\mathrm{d}\lambda(x) &= \int_G f(x)\overline{u(x)}\,\mathrm{d}\lambda(x) = \int_{\widehat{G}} \widehat{f}(\gamma)\overline{\widehat{u}(\gamma)}\,\mathrm{d}\omega(\gamma) \\ &= \int_{\widehat{G}} \widehat{f}(\gamma)\widehat{g}(\gamma_0\gamma^{-1})\,\mathrm{d}\omega(\gamma) = \widehat{f} * \widehat{g}(\gamma_0).\end{aligned}$$

Hence $f, g \in L^2(G)$ implies $\widehat{fg}(\gamma_0) = \widehat{f} * \widehat{g}(\gamma_0)$. Applying Plancherel's theorem, and using the surjectivity of the Fourier transform, for $F_1, F_2 \in L^2(\widehat{G})$ we find $f_1, f_2 \in L^2(G)$ such that $\widehat{f_i} = F_i$. We obtain

$$(\widehat{f_1 f_2})(\gamma_0) = F_1 * F_2(\gamma_0) \in A(\widehat{G}).$$

Conversely for $h \in L^1(G)$, write $h(x) := |h(x)| \cdot \psi(x)$ where

$$\psi(x) = \begin{cases} h(x)/\,|h(x)| & \text{if } h(x) \neq 0 \\ 1 & \text{otherwise.}\end{cases}$$

Define $f(x) = \sqrt{|h(x)|}$ and $g(x) = f(x)\psi(x)$. Then $f, g \in L^2(G)$, $h = f \cdot g$ and $\widehat{h} = \widehat{f} * \widehat{g}$ therefore $A(\widehat{G}) \subseteq \left\{F_1 * F_2 : F_2 \in L^2(\widehat{G})\right\}$, therefore concluding the proof. □

The following consequence will be used in the Pontryagin duality theorem.

Proposition 9.17. Let $E \subset \widehat{G}$ be non-empty and open. Then there is $\psi \in A(\widehat{G})$ with $\psi \neq 0$ and $\psi|_{\widehat{G}\setminus E} = 0$.

Proof. Pick any $\gamma_0 \in E$, and using the continuity at $(\gamma_0, e) \in \widehat{G} \times \widehat{G}$ of the product map we can find compact neighborhoods $K \ni \gamma_0$ and $F \ni e$ with $KF \subset E$. Then

$$\psi = \mathbf{1}_K * \mathbf{1}_F \in A(\widehat{G}), \quad \psi \neq 0.$$

However, $\operatorname{supp}\psi \subset KF \subset E$, which means that the restriction of ψ to $\widehat{G} \setminus E$ must be 0. □

9.4 The Pontryagin Duality Theorem

Let G be a locally compact abelian group, with $\widehat{G}$ locally compact abelian as well. Hence, $\widehat{\widehat{G}}$ is locally compact abelian, and everything we have proved about the pair $(G, \widehat{G})$ holds for $(\widehat{G}, \widehat{\widehat{G}})$. We have seen that

$$\begin{aligned} G \times \widehat{G} &\longrightarrow \mathbb{T} \\ (x, \gamma) &\longmapsto \gamma(x)\end{aligned}$$

is continuous, hence for all $x \in G$,

$$\begin{aligned} \widehat{G} &\longrightarrow \mathbb{T} \\ \gamma &\longmapsto \gamma(x)\end{aligned}$$

is a continuous character on $\widehat{G}$, which we denote by $\alpha(x) \in \widehat{\widehat{G}}$. Therefore
$$(x, \gamma) = (\gamma, \alpha(x)).$$

Theorem 9.18. The map $\alpha\colon G \longrightarrow \widehat{\widehat{G}}$ is an isomorphism of locally compact abelian groups.

Proof. First, we prove it is a homomorphism. For that,
$$(\gamma, \alpha(x_1x_2)) = (x_1x_2, \gamma) = (x_1, \gamma)(x_2, \gamma) = (\gamma, \alpha(x_1))(\gamma, \alpha(x_1)).$$
Therefore, for all $\gamma \in \widehat{G}$, $\alpha(x_1x_2) = \alpha(x_1)\alpha(x_2)$.

Now we prove injectivity. Let $x_1 \neq x_2$, by Corollary 9.12, $\widehat{G}$ separates points in G. Hence there exists $\gamma \in \widehat{G}$ such that $(x_1, \gamma) \neq (x_2, \gamma)$, that is, $(\gamma, \alpha(x_1)) \neq (\gamma, \alpha(x_2))$. Therefore $\alpha(x_1) \neq \alpha(x_2)$.

Now we prove that $\alpha\colon G \longrightarrow \alpha(G) \subset \widehat{\widehat{G}}$ is a homeomorphism onto its image, where of course $\alpha(G)$ has the topology induced by $\widehat{\widehat{G}}$. For every $C \subset \widehat{G}$ and $r > 0$, let
$$N_G(C, r) = \{x \in G : \ |(x, \gamma) - 1| < r, \quad \forall \gamma \in C\}.$$
By Corollary 9.11, the family of $N_G(C, r)$ for $C \subset \widehat{G}$ compact and $r > 0$ is a basis of open neighbourhoods of e in G. Now,
$$N_{\widehat{\widehat{G}}}(C, r) = \left\{\chi \in \widehat{\widehat{G}} : \ |(\gamma, \chi) - 1| < r, \quad \forall \gamma \in C\right\} \ni \widehat{\widehat{e}},$$
which follows from the continuity of the duality. By Proposition 7.3 these constitute open neighbourhoods of $\widehat{\widehat{e}}$.

Now, let us compute what $\alpha(N_G(C, r))$ looks like. If $x \in N_G(C, r)$, then $|(x, \gamma) - 1| < r$, which is the same as saying $|(\gamma, \alpha(x)) - 1| < r$, and that is $\alpha(x) \in N_{\widehat{\widehat{G}}}(C, r)$, since it holds for all $\gamma \in C$. This implies that
$$\alpha(N_G(C, r))) = N_{\widehat{\widehat{G}}}(C, r) \cap \alpha(G).$$
That is, $\alpha\colon G \longrightarrow \alpha(G)$ is continuous at e, and $\alpha^{-1}\colon \alpha(G) \longrightarrow G$ is continuous at $\widehat{\widehat{e}}$. Now, since they are continuous at the identity, they are continuous everywhere, meaning that α is a homeomorphism onto its image. Hence $\alpha(G) \subset \widehat{\widehat{G}}$ is locally compact.

Fact from topology: *If X is locally compact Hausdorff, then a subspace $Y \subset X$ is locally compact if and only if it is open in $\overline{Y}$.* We are going to use one of these directions.

Hence $\alpha(G)$ is an open subgroup of $\overline{\alpha(G)}$. Thus, $\alpha(G)$ is closed in $\overline{\alpha(G)} \subset \widehat{\widehat{G}}$, which implies that $\alpha(G)$ is closed in $\widehat{\widehat{G}}$, meaning that it is equal to its closure, $\alpha(G) = \overline{\alpha(G)}$. Now, if $\alpha(G) \neq \widehat{\widehat{G}}$, by Proposition 9.17, we can find $\psi \in A(\widehat{\widehat{G}})$ with $\psi|_{\alpha(G)} = 0$, $\psi \neq 0$, and $\psi = \widehat{f}$ for some $f \in L^1(\widehat{G})$. We have $\forall x \in G$, $\widehat{f}(\alpha(x)) = \psi(\alpha(x)) = 0$, so
$$\widehat{f}(\alpha(x)) = \int_{\widehat{G}} f(\gamma)\overline{(\gamma, \alpha(x))}\, \mathrm{d}\omega(\gamma) = \int_{\widehat{G}} f(\gamma)\overline{(x, \gamma)}\, \mathrm{d}\omega(\gamma) = 0$$
for every x. By the injectivity part of Corollary 9.7, $f = 0$ and thus $\psi = 0$, which yields a contradiction. □

9.5 Some consequences of the duality theorem

We will now explore some consequences of this duality theorem. Given a complex measure $\mu \in \mathcal{M}(G)$ on G we can define its Fourier transform as

$$\widehat{\mu}(\gamma) = \int_G \overline{(x,\gamma)}\, \mathrm{d}\mu(x).$$

This extends the Fourier transform on $L^1(G)$ via the identification of $L^1(G)$ with a subspace of $\mathcal{M}(G)$ by

$$\begin{array}{ccc} L^1(G) & \longrightarrow & \mathcal{M}(G) \\ f & \longmapsto & f \cdot \lambda, \end{array}$$

where λ is a fixed Haar measure on G. By duality, all the theorems that hold for $(G, \widehat{G})$ hold for $(\widehat{G}, G)$ as well. The next result follows for instance from Corollary 9.7.

Corollary 9.19. The map

$$\begin{array}{ccc} \mathcal{M}(G) & \longrightarrow & B(\widehat{G}) \\ \mu & \longmapsto & \widehat{\mu} \end{array}$$

is a $\mathbb{C}$-linear bijection.

Using the notation of Corollary 9.7, $\tilde{\mu}(\gamma) = \widehat{\mu}(\gamma^{-1})$, from which the result follows. Another example of this phenomenon is the following corollary.

Corollary 9.20. Let $\mu \in \mathcal{M}(G)$ and assume $\widehat{\mu} \in L^1(\widehat{G})$. Then there is $f \in L^1(G)$ such that $\mu = f \cdot \lambda$ and

$$f(x) = \int_{\widehat{G}} \widehat{\mu}(\gamma)(x,\gamma)\, \mathrm{d}\omega(\gamma)$$

for almost every $x \in G$.

Proof. By the hypotheses (see Corollary 9.19) $\widehat{\mu} \in L^1(\widehat{G}) \cap B(\widehat{G})$. Define

$$f(x) = \int_{\widehat{G}} \widehat{\mu}(\gamma)(x,\gamma)\, \mathrm{d}\omega,$$

then by the inversion theorem we have $f \in L^1(G)$ and

$$\widehat{\mu}(\gamma) = \int_G f(x)\overline{(x,\gamma)}\, \mathrm{d}\lambda(x).$$

But by definition it also holds that $\widehat{\mu}(\gamma) = \int_G \overline{(x,\gamma)}\, \mathrm{d}\mu(x)$. The uniqueness result in Corollary 9.19 now implies that $\mu = f \cdot \lambda$. $\square$

A very useful corollary is the following version of the inversion theorem.

Corollary 9.21. Assume $f \in L^1(G) \cap C(G)$ and assume $\widehat{f} \in L^1(\widehat{G})$. Then

$$f(x) = \int_{\widehat{G}} \widehat{f(\gamma)}(x,\gamma)\, \mathrm{d}\omega(\gamma), \quad \forall x \in G.$$

The next application concerns the following situation. Let $H < G$ be a closed subgroup of G. There is a canonical projection $p\colon G \longrightarrow G/H$, when G/H is endowed with the quotient topology. In this way, G/H is locally compact abelian Hausdorff, and p is a continuous open homomorphism. Another operation one may consider is, given H, to define

$$H^\perp = \left\{\gamma \in \widehat{G} : \ (h,\gamma) = 1, \quad \forall h \in H\right\} < \widehat{G}.$$

This is also a closed subgroup.

Theorem 9.22. The following canonical isomorphisms hold true

$$\widehat{G/H} \simeq H^\perp, \quad \widehat{H} \simeq \widehat{G}/H^\perp.$$

Proof. The first assertion follows from the fact that the map

$$\begin{aligned} \widehat{G/H} &\longrightarrow H^\perp \\ \chi \ &\longmapsto \chi \circ p \end{aligned}$$

is an isomorphism. By Pontryagin duality we have, from the preceding fact, that

$$\widehat{\widehat{G}/H^\perp} = H^{\perp\perp}.$$

It remains to verify that $H^{\perp\perp}$ recovers H.

First, it is clear that $H \subset H^{\perp\perp}$. If equality were not true, then there is $x \in H^{\perp\perp} \setminus H$. Hence $p(x) \neq e$ in the quotient G/H. But then there is $\chi \in \widehat{G/H}$ with $\chi(p(x)) \neq 1$. However, $\chi \circ p \in H^\perp$, hence $\chi \circ p(x) = 1$ since $x \in H^{\perp\perp}$, which yields a contradiction. □

Another consequence we obtain is the following.

Corollary 9.23. Let $H < G$ be a closed subgroup. Then every continuous character on H extends to a continuous character on G.

Proof. Let $\phi \in \widehat{H}$. By Theorem 9.22 we may view ϕ as an element in $\widehat{G}/H^\perp$. Hence, there must exist an element $\chi \in \widehat{G}$ which projects onto ϕ under the canonical projection. So we have $\chi|_H = \phi$. □

An interesting application of Theorem 9.22 is the following result.

Corollary 9.24. Let $\Gamma < G$ be a discrete subgroup with G/Γ compact. Then $\Gamma^\perp < G$ is a discrete subgroup and $\widehat{G}/\Gamma^\perp$ is compact.

Proof. Since G/Γ is compact, $\widehat{G/\Gamma} \simeq \Gamma^\perp$ is discrete. Now, since $\widehat{G}/\widehat{\Gamma} \simeq$ and Γ is discrete, $\widehat{\Gamma}$ is compact and hence $\widehat{G}/\Gamma^\perp$ is compact. □

Example 9.25. $G = \mathbb{R}^n$, let $\langle \cdot, \cdot \rangle$ be the usual inner product on $\mathbb{R}^n$. We identify $\mathbb{R}^n \longrightarrow \widehat{\mathbb{R}^n}$, $v \mapsto \chi_v$ where $\chi_v(w) = e^{2\pi i \langle v, w\rangle}$. A discrete subgroup $\Lambda \subset \mathbb{R}^n$ with $\mathbb{R}^n/\Lambda$ compact is called ***lattice***. For example, if $v_1, \ldots, v_n$ form a basis of $\mathbb{R}^n$, then $\Lambda = \mathbb{Z}v_1 + \cdots \mathbb{Z}v_n$ is a lattice in $\mathbb{R}^n$. Moreover, one can show that every lattice in $\mathbb{R}^n$ is of this form. Using the identification of $\mathbb{R}^n$ with $\widehat{\mathbb{R}^n}$ we can write

$$\Lambda^\perp = \{v \in \mathbb{R}^n : \forall w \in \Lambda,\ \langle v, w\rangle \in \mathbb{Z}\} = \{v \in \mathbb{R}^n : \forall 1 \leq i \leq n,\ \langle v, v_i\rangle \in \mathbb{Z}\}.$$

This gives for example $(\mathbb{Z}^n)^\perp = \mathbb{Z}^n$.

Example 9.26. Recall that the ring of adeles $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$ is defined as the subset

$$\left\{ (x_\infty, x_p, ...) \in \mathbb{R} \times \prod_{p\in\mathbb{P}} \mathbb{Q}_p \text{ with } x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\text{'s} \right\}.$$

Endowed with an appropriate topology it is a locally compact ring. Recall that for $x \in \mathbb{Q}_p$ there is a convergent series representation

$$x = \sum_{k=m}^{\infty} a_k p^k, \quad m \in \mathbb{Z}.$$

Its fractional part is defined as

$$\{x\} \coloneqq \sum_{k=m}^{-1} a_k p^k.$$

Then

$$(x_\infty, x_p, ...) \longmapsto e^{2\pi i \{x_\infty\}} \prod_{p\in\mathbb{P}} e^{-2\pi i\{x_p\}}$$

defines a continuous character χ of the additive group $\mathbb{A}_\mathbb{Q}$. Then

(i) The map

$$\begin{aligned} \mathbb{A}_\mathbb{Q} &\longrightarrow \widehat{\mathbb{A}_\mathbb{Q}} \\ a &\longmapsto \chi_a, \end{aligned}$$

where $\chi_a(b) = \chi(ab)$ is an isomorphism of locally compact abelian groups.

(ii) Under this identification, $\mathbb{Q}^\perp = \mathbb{Q}$.

Thus Theorem 9.22 implies that the dual $\widehat{Q}$ of the discrete group is given by $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$, which is compact.

We turn to a further application that involves a closed subgroup $H < G$, its orthogonal $H^\perp < G^\perp$ and the Fourier transform for functions on G, namely the Poisson formula. We will present a special case thereof for $\Gamma < G$ discrete with G/Γ compact.

Definition 9.27. We say that a continuous $f : G \longrightarrow \mathbb{C}$ is uniformly summable over Γ if

$$\sum_{\gamma\in\Gamma} |f(g\gamma)|$$

converges uniformly for g on compact subsets of G.

We call $T_\Gamma f(\dot g) := \sum_{\gamma\in\Gamma} f(g\gamma)$ the resulting function, where $\dot g = g\Gamma$ indicates that the right hand side as a function of g is Γ-invariant. Then it is an exercise to check that

$$T_\Gamma f \in C(G/\Gamma).$$

Theorem 9.28 (*Weil's formula*). Given a Haar measure α on G/Γ, there is a unique Haar measure λ on G such that for all $f \in C(G) \cap L^1(G)$ uniformly summable over Γ,

$$\int_G f(g)\,\mathrm{d}\lambda(g) = \int_{G/\Gamma} \mathrm{d}\alpha(\dot g)(T_\Gamma f)\,(\dot g).$$

Let us observe that $C_{00}(G)$ is contained in the space of uniformly summable functions over Γ and for those, the theorem follows readily. The Poisson formula is then formulated as follows.

Theorem 9.29 (*Poisson's formula*). Let $\Gamma < G$ be discrete with G/Γ compact. Let α be the Haar measure on G/Γ with $\alpha(G/\Gamma) = 1$ and λ be the corresponding Haar measure on G. Let $f \in L^1(G)\cap C(G)$, uniformly summable over Γ and assume $\widehat{f}|_{\Gamma^\perp}$ is in $\ell^1(\Gamma^\perp)$. Then

$$\sum_{\gamma\in\Gamma} f(\gamma) = \sum_{\eta\in\Gamma^\perp} \widehat{f}(\gamma).$$

Proof. We are going to use the Fourier inversion formula in its form given by Corollary 9.21. Since $\alpha(G/\Gamma) = 1$, the normalized dual Haar measure on $\Gamma^\perp = \widehat{G/\Gamma}$ is just the counting measure. Given f as in the statement of the theorem, let us compute $\widehat{T_\Gamma f}(\chi)$, where $\chi \in \Gamma^\perp$ and $\widehat{\cdot}$ refers to the Fourier transform on G/Γ. We have

$$\widehat{T_\Gamma f}(\chi) = \int_{G/\Gamma} T_\Gamma f(\dot g) \cdot \overline{(\dot g, \chi)}\,\mathrm{d}\alpha(\dot g).$$

Now observe that

$$T_\Gamma f(\dot g)\overline{(\dot g, \chi)} = \sum_{\gamma\in\Gamma} f(g\gamma)\overline{(g\gamma, \chi)}$$

since $\chi \in \Gamma^\perp$. Hence by Weil's formula, we get

$$\widehat{T_\Gamma f}(\chi) = \widehat{f}(\chi).$$

Since $T_\Gamma f$ lies in $C(G/\Gamma) \subset L^1(G/\Gamma)$ and $L^1(\Gamma^\perp) \ni \widehat{f}|_{\Gamma^\perp} = \widehat{T_\Gamma f}$, the inversion formula applied at $\dot e \in G/\Gamma$ gives

$$T_\Gamma f(\dot e) = \sum_{\eta\in\Gamma^\perp} \widehat{T_\Gamma f}(\eta) = \sum_{\eta\in\Gamma^\perp} \widehat{f}(\eta).$$

Taking into account that

$$T_\gamma f(\dot e) = \sum_{\gamma\in\Gamma} f(\gamma)$$

establishes the formula. □

10 Wiener's theorem

The material on Regular Algebras is taken from [GeRaSh], 6.34 and 6.36.

The first form of Wiener's theorem is described in the following statement.

Theorem 10.1 (*Wiener*). Let $f \in L^1(G)$. Then the ideal $I = \left\{g * f : \ g \in L^1(G)\right\}$ generated by f is dense in $L^1(G)$ if and only if $\widehat{f}(\chi) \neq 0$ for every $\chi \in \widehat{G}$.

This admits a reformulation in terms of the translation invariance of $L^1(G)$ generated by f provided one shows the next proposition.

Proposition 10.2. A closed vector subspace of $L^1(G)$ is an ideal if and only if it is translation invariant.

An important consequence is the Wiener Tauberian theorem.

Theorem 10.3 (*Wiener Tauberian theorem*). Assume $\Phi \in L^\infty(G)$ and $f \in L^1(G)$ to be such that $\widehat{f}(\chi) \neq 0$ for every $\chi \in \widehat{G}$, and

$$f * \Phi(x) \longrightarrow a\widehat{f}(e), \quad \text{as} \quad x \to \infty.$$

Then

$$g * \Phi(x) \longrightarrow a\widehat{g}(e), \quad \text{as} \quad x \to \infty$$

holds for every $g \in L^1(G)$.

In fact Wiener's theorem will immediately follow from the following result.

Theorem 10.4. Le $I \subset L^1(G)$ be an ideal such that $\left\{\chi \in \widehat{G} : \ \widehat{f}(\chi) \neq 0 \quad \forall f \in I\right\}$ is empty. Then I is dense in $L^1(G)$.

10.1 Regular Banach algebras

In all this section, B will be a Banach algebra with identity e. Therefore $\widehat{B}$, its Guelfand spectrum, is a compact set.

Definition 10.5. A semisimple Banach algebra B is regular if for all $F \subsetneq B$ closed and $\chi \notin F$ there exists $a \in B$ with $\widehat{a}|_F = 0$ and $\widehat{a}(\chi) \neq 0$.

Given a closed subset $F \subset \widehat{B}$, we denote $I(F) = \{a \in B : \widehat{a}|_F = 0\}$ and given any ideal $I \subset B$, we let

$$Z(I) = \left\{\chi \in \widehat{B} : \widehat{a}(\chi) = 0 \quad \forall a \in I\right\}.$$

Observe that when B is regular,

$$Z(I(F)) = F$$

for every closed subset $F \subset \widehat{B}$. Our objective is to prove the next theorem.

Theorem 10.6. Let B be a regular Banach algebra and $F \subset \widehat{B}$ closed. Then among all ideals $I \subset B$ with $Z(I) = F$, there is a minimal one $J(F)$ given by

$$J(F) = \{a \in B : \widehat{a} \text{ vanishes in a neighborhood of } F\}.$$

For this we will need the following lemma.

Lemma 10.7. Let B be regular, $F \subset B$ closed. Then composing a non-zero character of $B/I(F)$ with the projection $B \longrightarrow B/I(F)$ identifies $\widehat{B/I(F)}$ with F.

Proof. Let $\chi \in F$. Then $\widehat{a}(\chi) = 0$ for every $a \in I(F)$ and hence χ factors via the projection $\mathrm{pr} : B \longrightarrow B/I(F)$. Conversely, let $\chi \in \widehat{B/I(F)}$. Then χ composed with the projection belongs in $\widehat{B}$. If it were not in F, then by regularity there exists $a \in B$ with $\widehat{a}|_F = 0$ and $\widehat{a}(\chi \circ \mathrm{pr}) \neq 0$, that is, $a \in I(F)$ and $\chi(\mathrm{pr}(a)) \neq 0$, which is a contradiction since $\mathrm{pr}(a) = 0$. □

Lemma 10.8. Let B be regular and $I \subset B$ any ideal and $F \subset \widehat{B}$ closed with $F \cap Z(I) = \emptyset$. Then there is $a \in I$ with $\widehat{a}|_F = 1$.

Proof. Consider the projection $\mathrm{pr} : B \longrightarrow B/I(F)$. The maximal ideals of $B/I(F)$ are given by $\mathrm{pr}(I\{\chi\})$, for $\chi \in F$ (see Lemma 10.7). By the hypotheses, $\mathrm{pr}(I)$ is not contained in $\mathrm{pr}(I\{\chi\})$, hence $\mathrm{pr}(I) = B/I(F)$. Therefore, in particular, there is $a \in I$ such that $\mathrm{pr}(a) = \mathrm{pr}(e)$, and $a - e \in I(F)$. Hence $\widehat{a}|_F = 1$. □

We can now tackle the proof of Theorem 10.6.

Proof of Theorem 10.6. Step 1. First, $Z(J(F)) = F$: indeed, pick $x \notin F$ and $O \supset F$ open with $x \notin \overline{O}$. Then by regularity we can find $a \in B$ with $a(x) \neq 0$ and $a|_{\overline{O}} = 0$, hence $a \in J(F)$.

Step 2. Let $I \subset B$ be any ideal with $Z(I) = F$. Let $a \in J(F)$ and define

$$F_1 = \left\{x \in \widehat{B} : \widehat{a}(\chi) = 0\right\} \supset F$$

and $F_2 := \overline{\widehat{B} \setminus F_1}$. Since $a \in J(F)$, we have $F_2 \cap F = \emptyset$ and by Lemma 10.8 there is $x \in I$ with $\widehat{x}|_{F_2} = 1$. Then $\widehat{x} \cdot \widehat{a} = \widehat{a}$ and hence by semisimplicity,

$$x \cdot a = a,$$

which implies $a \in I$ and hence $J(F) \subset I$. □

10.2 Applications to $L^1(G)$

The preceding results will be applied to $B = L^1(G) \oplus \mathbb{C} \cdot e$, so that $\widehat{B} = \beta\widehat{G}$ is the one point compactification of $\widehat{G}$.

Lemma 10.9. The algebra B is regular.

Proof. We already know that B is semisimple. For regularity, let $F \subset \widehat{B}$ be closed and $\chi_0 \notin F$. There are two cases. The first one is when $F \subset \beta\widehat{G}$ and $\chi_0 \neq \{\infty\}$. Then we know that there is $f \in L^1(G)$ with $\widehat{f}|_{F\cap\widehat{G}} = 0$ and $\widehat{f}(\chi_0) \neq 0$ (Proposition 9.17). In all cases we have that $\widehat{f}(\infty) = 0$.

The second case happens when $F \subset \widehat{G}$ and $\chi_0 = \infty$. Then F is compact. Let $V \subset \widehat{G}$ be open with $\overline{V}$ compact, and define

$$f(\chi) = \frac{\chi_{FV} * \chi_{V^{-1}}}{\omega(FV)}(\chi), \quad \forall \chi \in \widehat{G}.$$

Then $f \in A(\widehat{G})$ and $\forall \chi \in F$, $f(\chi) = 1$, which can be checked by simply integrating over V. Thus we find that $f = \widehat{u}$, for $u \in L^1(G)$ satisfying $\widehat{u}|_F = 1$. Then $u - e \in B$ satisfies $\widehat{u-e}|_F = 0$ and $\widehat{u-e}(\infty) = -1$. □

Lemma 10.10. The ideal

$$J(\infty) = \left\{ f \in L^1(G) : \ \widehat{f} \in C_{00}(\widehat{G}) \right\}$$

is dense in $L^1(G)$.

Proof. First, the subspace

$$X = \{ g \in L^2(G) : \ \widehat{g} \text{ is compactly supported} \}$$

is dense in $L^2(G)$ by Plancherel's theorem. Thus,

$$\{ v = gh : \ g, h \in X \}$$

is dense in $L^1(G)$. But $\widehat{v} = \widehat{g} * \widehat{h} \in C_{00}(\widehat{G})$, from which the lemma follows. □

Now we prove Theorem 10.4.

Proof of Theorem 10.4. Let $I \subset L^1(G)$ be an ideal with the stated assumption. Then $I \subset B$ is an ideal as well and Riemann-Lebesgue implies $Z(I) = \{\infty\}$. By Theorem 10.6,

$$I \supset J(\{\infty\})$$

and the latter is dense in $L^1(G)$ by Lemma 10.10. □

Now, we have showed Theorem 10.4, which immediately implies Theorem 10.1. In fact, the logic for Theorem 10.3 (Wiener's Tauberian formula) is to first establish Proposition 10.2 and then make it follow from Theorem 10.4. In order to prove Proposition 10.2, we use the Hahn-Banach theorem, which implies that if $I \subset L^1(G)$ is a closed subspace, then $I = \bigcap \ker L$, where the intersection is taken over all $L \in L^1(G)^*$ with ker $L \supset I$. Now let I be translation invariant and $\Phi \in L^\infty(G)$ such that ker $L_{\check{\Phi}} \supset I$, where

$$L_{\check{\Phi}}(f) = \int_G f(x)\Phi(x^{-1})\,\mathrm{d}\lambda(x).$$

Since with $f \in I$, $\lambda(y)f \in I$, it follows that

$$\ker L_{\check{\Phi}} \supset I \iff f * \Phi = 0, \quad \forall f \in I.$$

Proof of Proposition 10.2. If I is closed, translation invariant and ker $L_{\check{\Phi}} \supset I$, then $f * \Phi = 0$ for every $f \in I$. Hence, if $g \in L^1(G)$, $g * f * \Phi = 0$, from which it follows by evaluating at e that

$$L_{\check{\Phi}}(g * f) = 0.$$

Hence $g * f \in$ ker $L_{\check{\Phi}}$ and thus $g * f \in I$. Therefore, I is an ideal.

Let I be a closed ideal and $\Phi \in L^\infty(G)$ with $I \subset$ ker $L_{\check{\Phi}}$. Then, for any $g \in L^1(G)$, $f * g \in I$ and

$$0 = L_{\check{\Phi}}(f * g) = f * g * \Phi(e) = f * \Phi * g(e),$$

that is

$$\int_G (f * \Phi)(x)\check{g}(x)\,\mathrm{d}\lambda(x) = 0$$

for every $g \in L^1(G)$ and for the bounded continuous function $f * \Phi$ we have $f * \Phi = 0$. This, however, can be rewritten as

$$L_{\check{\Phi}}\big(\lambda(y^{-1})f\big) = 0, \quad \forall y \in G,$$

meaning that $\lambda(y)f \in$ ker $L_{\check{\Phi}}$. This implies that $\lambda(y)f \in I$ for every $y \in G$, and thus I is translation invariant. □

We complete the discussion with the proof of Theorem 10.3.

Proof of Theorem 10.3. Observe that the hypothesis

$$f * \Phi(x) \longrightarrow a\widehat{f}(e), \quad x \to \infty$$

can be written as

$$f * (\Phi - a\mathbb{1})(x) \longrightarrow 0, \quad x \to \infty.$$

Thus, we may assume $a = 0$, and hence $f * \Phi \in C_0(G)$. Consider

$$I := \left\{g \in L^1(G) : \ g * \Phi \in C_0(G)\right\}.$$

Clearly I is translation invariant since $C_0(G)$ is. Moreover, the map $g \mapsto g * \Phi$ defined on $L^1(G)$ takes its values in the space $C^b(G)$ of continuous bounded functions, and

$$\|g * \Phi\|_\infty \leq \|g\|_1 \|\Phi\|_\infty,$$

that is, it is a bounded operator. Since $C_0(G)$ is closed in $C^b(G)$ for the L^∞-norm, this implies that I is closed and translation invariant, hence a closed ideal in $L^1(G)$. Since $f \in I$ and $\widehat{f}(\chi) \neq 0$ for all $\chi \in \widehat{G}$, this, together with Theorem 10.4, implies that $I = L^1(G)$. □

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