Algebraic Topology

Lecture notes for a one semester course taught by professor Will Merry

Contents include homotopy, homology and the fundamental group.

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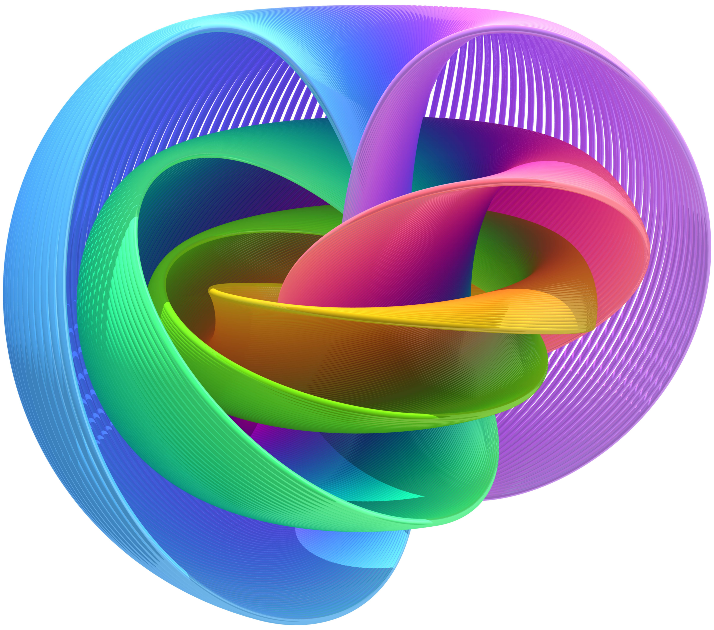

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Introduction

Most of modern mathematics can be regarded as classification problems. These are problems in which $\mathfrak{X}$ represents a collection of mathematical objects, and X, Y are two elements of $\mathfrak{X}$, and the question is how can one tell if X and Y are actually the same object.

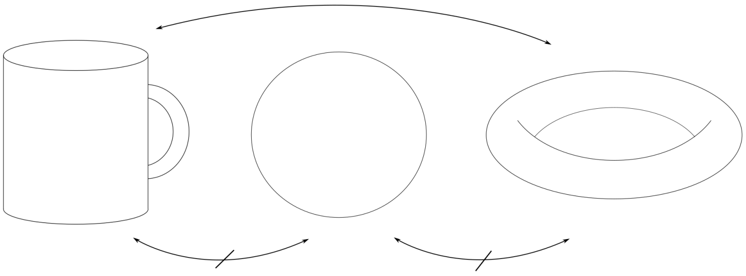
The first example of this is when $\mathfrak{X} = \mathbb{Z}$. If $m, n \in \mathbb{Z}$, is $m = n$? Of course, this example is devoid of content. Taking $\mathfrak{X} = \mathbb{Z}/3\mathbb{Z}$, and asking whether $m = n$ is equivalent to asking whether $x = y$ in $m = 4k + x, n = 3l + y$. This, again, is not particularly interesting. However, taking $m = 17846194845$ and $n = 845865234102$ makes the question somewhat less subtle. These problems are not very difficult, and the reason for this is that $\mathfrak{X}$ is not particularly interesting.

By enriching the structure of $\mathfrak{X}$ we can obtain slightly more interesting examples. For instance, let $M, N \in \mathbb{M}_{n\times n}(\mathbb{R})$. How can we determine if $\det M = \det N$. For $n = 2$ this is particularly easy, and if $n > 2$ there are ways to compute $\det M$ and $\det N$. If we take $\mathfrak{X}$ to be the collection of all groups and let G, H be two groups in $\mathfrak{X}$, the question is how can we determine when G and H are isomorphic. This is generally not easy, particularly when the presentation of the groups is inconvenient. For example,

$$\begin{aligned} G &= \langle a, b, c : \ c = aba^{-1}b^{-1}, ac = ca, bc = cb\rangle, \\ H &= \langle a, b, c : \ aba = bab, (aba)^4 = c^2 = (ca)^2 = (bc)^2 = 0\rangle. \end{aligned}$$

In this case H and G are not isomorphic.

In topology, the question is whether one topological space can be turned into another without tearing it.



These problems are interesting and generally hard, as $\mathfrak{X}$ is more complex. Algebraic topology aims to turn topological classification problems into algebraic classification problems, which are comparatively easier.

Contents

1 The Brouwer Fixed Point Theorem

Let us illustrate the philosophy behind algebraic geometry by looking at the example of Brouwer's fixed point theorem. We introduce some notation that will be used throughout the course.

Notation. We will use B^n to denote the closed unit ball in $\mathbb{R}^n$,

$$B^n = \{x \in \mathbb{R}^n : \; ||x|| \leq 1\}\,.$$

The boundary of B^n is the $(n-1)$–dimensional unit sphere $\mathbb{S}^{n-1}$,

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \; ||x|| = 1\}\,.$$

Theorem 1.1 (*Brouwer's fixed point theorem*). For all $n \geq 1$, every continuous map $f : B^n \longrightarrow B^n$ has a fixed point.

Proof. For the case $n = 1$ the proof is easy. Set $f(-1) = a$, $f(1) = b$, and assume $a > -1$ and $b < 1$ without loss of generality, as otherwise either 1 or -1 would be fixed points. Let

$$\begin{aligned} \text{Gr}(f) &= \{(x, f(x)) : \; x \in (-1, 1)\} \\ \Delta &= \{(x, x) : \; x \in (-1, 1)\}\,. \end{aligned}$$

Note that a fixed point of f corresponds to an intersection $\text{Gr}(f) \cap \Delta$. Now define

$$\begin{aligned} A &= \{(x, f(x)) : f(x) > x\} \\ B &= \{(x, f(x)) : f(x) < x\}\,. \end{aligned}$$

Both A and B are nonempty, open and disjoint, since $(-1, a) \in A$, $(1, b) \in B$ and f is continuous. If $\text{Gr}(f) \cap \Delta = \emptyset$ then $\text{Gr}(f) = A \cup B$, but since f is continuous, $\text{Gr}(f)$ is connected, and this contradicts $\text{Gr}(f) = A \cup B$. $\square$

Interestingly enough, for $n > 1$ there is no known simple proof. A "pleasing" proof uses the tools of algebraic topology. We will construct a homology functor H_n for each $n \geq 0$ that turns topological spaces into abelian groups, and continuous maps into group homomorphisms,

$$\begin{aligned} H_n : \quad X \quad &\longmapsto \quad H_n(X) \\ H_n : f : X \to Y &\longmapsto H_n(f) : H_n(X) \to H_n(Y)\,. \end{aligned}$$

Also, it satisfies the following properties for all $n \geq 1$.

(i) $H_n(g \circ f) = H_n(g) \circ H_n(f)$.

(ii) $H_n(id_X) = id_{H_n(X)}$.

(iii) $H_n(B^{n+1}) = \{0\},\ H_n(\mathbb{S}^n) \neq \{0\}$.

For now we will accept such H_n exist, and later in the course we will prove it and use them frequently.

Definition 1.2. Let X be a subspace of the topological space Y. We say that X is a retract of Y if there exists a continuous map $r : X \longrightarrow Y$ such that if $i : X \hookrightarrow Y$ denotes the inclusion map, then the following diagram commutes.

$$\begin{tikzcd}
 & Y \arrow[dr, "r"] & \\
X \arrow[ur, "i"] \arrow[rr, "id_X"] & & X
\end{tikzcd}$$

We say r is a retraction.

We will now prove that $\mathbb{S}^n$ is not a retract of B^{n+1}.

Lemma 1.3. $\mathbb{S}^n$ is not a retract of B^{n+1}.

Proof. Suppose such an r exists. Then, using the properties of H_n, the following diagram commutes.

$$\begin{tikzcd}
 & H_n(B^{n+1}) \arrow[dr, "H_n(r)"] & \\
H_n(\mathbb{S}^n) \arrow[ur, "H_n(i)"] \arrow[rr, "id_{H_n(\mathbb{S}^n)}"] & & H_n(\mathbb{S}^n)
\end{tikzcd}$$

However, $H_n(B^{n+1}) = \{0\}$ and $id_{H_n(\mathbb{S}^n)}$ cannot be factored through $r \equiv 0$ since $H_n(\mathbb{S}^n) \neq \{0\}$. ☐

With this result we can formulate the proof for Brouwer's fixed point theorem when $n \geq 1$.

Proof (Brouwer's fixed point theorem). Suppose $f : B^{n+1} \longrightarrow B^{n+1}$ has no fixed points. Then we can define $r(x)$ as the point where the ray that originates from x and goes through $f(x)$ intersects $\mathbb{S}^n$. The function r is a retraction and it is continuous since f is. This contradicts the lemma.

☐

We will now explore the basics of category theory and explain what functors are.

1.1 Category Theory

Category theory is a way to formulate mathematical theories in an abstract and general way to prove results in many areas of mathematics. It aims to generalize the idea of structure, and thus applies to algebra, analysis, set theory, etc. Categories are defined as "collections" of mathematical objects. Sets, groups, vector spaces, topological spaces, manifolds, etc. can be thought of as categories. Functions that preserve structure between these objects are called morphisms. Some examples of morphisms include functions, group homomorphisms, linear functions, continuous functions, etc.

Definition 1.4. A category $\mathcal{C}$ consists of the following data.

(i) A class of objects, $\operatorname{obj}\mathcal{C}$.

(ii) Given two objects A, B of $\mathcal{C}$, there is a set of morphisms, $\operatorname{Hom}_{\mathcal{C}}(A, B)$ from A to B.

(iii) There is an operation, "$\circ$", defined as

$$\begin{aligned} \circ : \operatorname{Hom}_{\mathcal{C}}(A,B) \times \operatorname{Hom}_{\mathcal{C}}(B,C) &\longrightarrow \operatorname{Hom}_{\mathcal{C}}(A,C) \\ (f,g) &\longmapsto g \circ f, \end{aligned}$$

which satisfies the following properties.

- For all $A, B, C, D \in \operatorname{obj}\mathcal{C}$ and $\forall f \in \operatorname{Hom}_{\mathcal{C}}(A,B), \forall g \in \operatorname{Hom}_{\mathcal{C}}(B,C), \forall h \in \operatorname{Hom}_{\mathcal{C}}(C,D)$,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- For each $A \in \operatorname{obj}\mathcal{C}$ there is a morphism $id_A \in \operatorname{Hom}_{\mathcal{C}}(A,A)$ such that
$$\begin{aligned} f \circ id_A &= f, \\ id_A \circ g &= g. \end{aligned}$$

We will now define what functors are.

Definition 1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $T : \mathcal{C} \longrightarrow \mathcal{D}$ consists of the following data.

(i) For each $A \in \operatorname{obj}\mathcal{C}$, $T(A) \in \operatorname{obj}\mathcal{D}$.

(ii) For each $f \in \operatorname{Hom}_{\mathcal{C}}(A,B)$, $T(f) \in \operatorname{Hom}_{\mathcal{D}}(T(A), T(B))$ such that

$$\begin{aligned} T(g \circ_{\mathcal{C}} f) &= T(g) \circ_{\mathcal{D}} T(f), \\ T(id_A) &= id_{T(A)}. \end{aligned}$$

Let us now take a look at several examples of functors. The first one is a "forgetful functor". Let us define

$$\begin{aligned} T : \quad \text{Top} &\longrightarrow \text{Sets} \\ X &\longmapsto X \\ f : X \to Y &\longmapsto f : X \to Y. \end{aligned}$$

The functor T essentially forgets about the topological structure of the space X, as in particular it is a set, and T regards continuous functions as functions between sets. Every time we have a couple of structures in which one is strictly richer than the other (that is it is built up on the weaker structure) we can define a forgetful functor like T by regarding the richer structure as one example of the weaker one. Also, some structures naturally induce other ones. For example, finite dimensional, real vector spaces induce a canonical topology, which makes them able to be regarded as topological spaces. Thus, we can define a forgetful functor from the category of vector spaces which regards vector spaces as topological spaces (with the canonical topology) and linear functions as continuous functions.

Later in the course we will define the homology functors, which we used previously for proving Brouwer's fixed point theorem. These are functors from the category of topological spaces into the category of abelian groups, which is a subcategory of the category of groups. Another example of a functor that turns topological spaces into groups is the fundamental group

$$\pi_1 : \mathrm{Top} \longrightarrow \mathrm{Groups}\,.$$

2 Homotopy

Homotopy is the mathematical way of expressing deformation (of functions or topological spaces) without tearing.

Remark. Throughtout the course we will use the terms "function" and "continuous function" indistinctly since we are talking about the category Top and the elements of $\mathrm{Hom}_{\mathrm{Top}}(A, B)$ are continuous functions. Also, we will use some notation for the unit interval, namely $I = [0, 1]$ and $\partial I = \{0, 1\}$.

Definition 2.1. Suppose X, Y are topological spaces and $f_0, f_1 : X \longrightarrow Y$ are continuous. A homotopy F from f_0 to f_1 is a continuous map $F : X \times I \longrightarrow Y$ such that

$$\begin{aligned} F(x, 0) &= f_0(x), \\ F(x, 1) &= f_1(x). \end{aligned}$$

Defining

$$\begin{aligned} f_t : X &\longrightarrow Y \\ x &\longmapsto F(x, t), \end{aligned}$$

we obtain a family $\{f_t\}$ of continuous functions which deforms f_0 into f_1 and depends continuously on t. We write $F : f_0 \simeq f_1$ to indicate that F is a homotopy from f_0 to f_1, and we write $f_0 \simeq f_1$ to indicate that such a homotopy exists. The following lemma will prove very useful for proving results throughout the course. For this reason we will give it a name, the "gluing lemma".

Lemma 2.2 (*Gluing lemma*). Let X be a topological space and suppose we can write

$$X = \bigcup_{i=1}^{k} X_i,$$

where X_i is a closed subspace for every i. Let $f_i : X_i \longrightarrow Y$ be continuous and suppose that for all i, j, if $X_i \cap X_j \neq \emptyset$, then $f_i|_{X_i \cap X_j} \equiv f_j|_{X_i \cap X_j}$. Then $\exists! f : X \longrightarrow Y$ such that $f|_{X_i} = f_i \quad \forall i$.

Proof. Uniqueness is clear. We need only show that f is continuous. Let $C \subset Y$ be a closed set. We show that $f^{-1}(C)$ is closed in X.

$$f^{-1}(C)=\left(\bigcup_{i=1}^{k} X_i\right)\cap f^{-1}(C)=\bigcup_{i=1}^{k}\left(X_i\cap f^{-1}(C)\right)=\bigcup_{i=1}^{k}\left(X_i\cap f_i^{-1}(C)\right).$$

□

Although we formulated the lemma with closed sets for our purpose, there is an alternative formulation in terms of open sets. Suppose $X=\bigcup_{i=1}^{k} X_i$ where each X_i is open. The rest of the formulation follows as above. Our first application of the gluing lemma will be to show that homotopy is an equivalence relation on the space of continuous functions between two topological spaces.

Proposition 2.3. Let X, Y be topological spaces. Then the relation $\simeq$ of homotopy on $C(X,Y)$ is an equivalence relation.

Proof. First, we check that $f\simeq f$ by $F(x,t)=f(x)$. We now suppose $f\simeq g$. This means that $\exists F$ such that $F(x,0)=f(x)$ and $F(x,1)=g(x)$. Define $G(x,t)=F(x,1-t)$. G is continuous since it's the composition of continuous functions. Thus, $g\simeq f$. We are left to prove transitivity. Suppose $f\simeq g$ and $g\simeq h$. Define

$$\begin{aligned} H: X\times I &\longrightarrow Y \\ (x,t) &\longmapsto \begin{cases} F(x,2t) & t\in[0,1/2] \\ G(x,2t-1) & t\in[1/2,1]. \end{cases}\end{aligned}$$

Then H is continuous by the gluing lemma. □

Definition 2.4. Given $f\in C(X,Y)$ we denote by $[f]$ its equivalence class. Thus $g\in[f]$ means that there exists $F: f\simeq g$.

Lemma 2.5. Composition respects equivalence classes. Suppose

$$\begin{aligned} f_0,f_1 &: X\longrightarrow Y \\ g_0,g_1 &: Y\longrightarrow Z. \end{aligned}$$

Assume $[f_0]_{X,Y}=[f_1]_{X,Y}$ and $[g_0]_{Y,Z}=[g_1]_{Y,Z}$. Then

$$[g_0\circ f_0]_{X,Z}=[g_1\circ f_1]_{X,Z}.$$

Proof. Let $F: f_0\simeq f_1$ and $G: g_0\simeq g_1$. Define a map

$$\begin{aligned} H: X\times I &\longrightarrow Z \\ (x,t) &\longmapsto G(f_0(x),t). \end{aligned}$$

Now we check,

$$\begin{aligned} H(x,0)&=G(f_0(x),0)=g_0\circ f_0(x), \\ H(x,1)&=G(f_0(x),1)=g_1\circ f_0(x). \end{aligned}$$

Define

$$\begin{aligned} K : X \times I &\longrightarrow \quad Z \\ (x,t) &\longmapsto g_1\left(F(x,t)\right). \end{aligned}$$

Again, we check,

$$\begin{aligned} K(x,0) &= g_1\left(F(x,0)\right) = g_1 \circ f_0(x), \\ K(x,1) &= g_1\left(F(x1)\right) = g_1 \circ f_1(x). \end{aligned}$$

Now,

$$\begin{aligned} H &: g_0 \circ f_0 \simeq g_1 \circ f_0, \\ K &: g_1 \circ f_0 \simeq g_1 \circ f_1. \end{aligned}$$

□

2.1 Isomorphisms

Now let us go back to category theory to define what isomorphisms are and how their role in the category of topological spaces does not help very much with classification problems.

Definition 2.6. Let $\mathcal{C}$ be a category. An isomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is a morphism with the property that $\exists g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ such that

$$\begin{aligned} g \circ f &= id_A, \\ f \circ g &= id_B. \end{aligned}$$

Some examples of isomorphisms are bijections in the category of sets, group isomorphisms in the category of groups, homeomorphisms in the category of topological spaces, vector isomorphisms in the category of vector spaces, etc. In general, there is no reason for isomorphisms to exist between any two objects in a category.

Remark. The notions of monomorphisms (injective) and epimorphisms (surjective) are more complicated to define in the framework of category theory.

When trying to solve classification problems in topology, one would think studying the isomorphisms in the category of topological spaces would be the key to succeed. However, homeomorphisms are too rigid. It is not the right choice to look at them. However, it is much better to look at the equivalence classes of homotopy. For that reason, we will now explain how to take quotients on categories.

2.2 Quotient categories

Definition 2.7. Let $\mathcal{C}$ be a category. A congruence on $\mathcal{C}$ is an equivalence relation $\sim$ on

$$\bigcup_{(A,B)\in \mathrm{obj}\,\mathcal{C}\times\mathrm{obj}\,\mathcal{C}} \mathrm{Hom}(A,B)$$

which satisfies

(i) If $f \in \mathrm{Hom}(A,B)$ and $f \sim g$ then $g \in \mathrm{Hom}(A,B)$.

(ii) If $f_0 \sim f_1$, $g_0 \sim g_1$ and $g_0 \circ f_0$ and $g_1 \circ f_1$ are defined, then $g_0 \circ f_0 \sim g_1 \circ f_1$.

A congruence allows us to construct a quotient category. This is the category $\mathcal{C}/\sim$ such that

- $\operatorname{obj}(\mathcal{C}/\sim) = \operatorname{obj}(\mathcal{C})$.
- $\operatorname{Hom}_{\mathcal{C}/\sim}(A, B) = \{[f] : f \in \operatorname{Hom}_{\mathcal{C}}(A, B)\}$.
- Composition:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}/\sim}(A, B) \times \operatorname{Hom}_{\mathcal{C}/\sim}(B, C) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}/\sim}(A, C) \\ ([f], [g]) & \longmapsto & [g \circ f]. \end{array}$$

The definition of morphisms in the quotient category makes sense by (i) and the definition of composition makes sense by (ii), since if $f_0, f_1 \in [f]$ and $g_0, g_1 \in [g]$ then $[g_0 \circ f_0] = [g_1 \circ f_1]$.

Theorem 2.8. Homotopy is a congruence on Top.

This result follows from lemma 2.5, since it is an equivalence relation which is respected by composition.

Definition 2.9. hTop is the quotient category Top $/ \simeq$, the homotopy topological category.

What are isomorphisms in quotient categories? Let $\mathcal{C}$ be a category and $\sim$ a congruence on $\mathcal{C}$. Let $[f] \in \operatorname{Hom}_{\mathcal{C}/\sim}(A, B)$. Then by definition, $[f]$ is an isomorphism if and only if there is $[g] \in \operatorname{Hom}_{\mathcal{C}/\sim}(B, A)$ such that

$$[g] \circ [f] = id_{\operatorname{Hom}_{\mathcal{C}/\sim}(A,A)},$$
$$[f] \circ [g] = id_{\operatorname{Hom}_{\mathcal{C}/\sim}(B,B)}.$$

Which is the same as $g \circ f \sim id_A$ and $f \circ g \sim id_B$. Explicitly in hTop, this means that $f :\longrightarrow Y$ is an isomorphism if there exists $g : Y \longrightarrow X$ such that $f \circ g \simeq id_X$ and $g \circ f \simeq id_Y$. Such maps $f : X \longrightarrow Y$ are known as homotopy equivalences. That is, $f : X \longrightarrow Y$ is a homotopy equivalence if $[f]$ is an isomorphism in hTop.

Definition 2.10. Two topological spaces X and Y have the same homotopy type or are said to be homotopy equivalent if there exists a homotopy equivalence $f : X \longrightarrow Y$.

We can now formulate the classification problem in topology:

Let X, Y be topological spaces. Are they homotopy equivalent?

We will spend the rest of this course failing to answer this question. We now look at the opposite extreme. Clearly, homeomorphic spaces are homotopy equivalent, but the converse is not true. We will see examples of this in the future.

Definition 2.11. Let X, Y be topological spaces. A constant map $c : X \longrightarrow Y$ is a map with $c(X) = y$ for some constant $y \in Y$. A map $f : X \longrightarrow Y$ is said to be nullhomotopic if f is homotopic to some constant map.

When $X = \mathbb{S}^n$, there is an easy criterion for deciding whether a map is nullhomotopic. Before stating the result, we need one more definition.

Definition 2.12. Suppose X, Y are topological spaces and Z is a subset of X. Suppose $f_0, f_1 : X \longrightarrow Y$ are such that $f_0|_Z \equiv f_1|_Z$. We say that f_0 and f_1 are homotopic relative to Z, $f_0 \simeq f_1 \text{ rel } Z$, if $\exists F : f_0 \simeq f_1$ such that

$$F(x,t) = f_0(x) = f_1(x) \quad \forall (x,t) \in Z \times I.$$

For fixed X, being homotopic relative to Z is an equivalence relation. For the case in which $Z = \{p\}$ is a single point, we will write "rel p" instead of "rel $\{p\}$".

Proposition 2.13. Let Y be a topological space and $n \geq 0$. Let $f : \mathbb{S}^n \longrightarrow Y$ be a continuous function. The following are equivalent.

(i) f is nullhomotopic.

(ii) $\exists g : B^{n+1} \longrightarrow Y$ continuous such that $g|_{\mathbb{S}^n} = f$.

(iii) Fix $p \in \mathbb{S}^n$ and let c denote the constant map

$$\begin{aligned} c : \mathbb{S}^n &\longrightarrow Y \\ x &\longmapsto f(p), \end{aligned}$$

then $f \simeq c \text{ rel } p$.

Proof. (iii) $\Longrightarrow$ (i) is trivial. For (i) $\Longrightarrow$ (ii), let c be a constant map such that $F : f \simeq c$. Say $c(x) = q$ for all $x \in \mathbb{S}^n$. Define $g : B^{n+1} \longrightarrow Y$ by

$$g(x) = \begin{cases} q & ||x|| \leq 1/2 \\ F\left(\frac{x}{||x||}, 2 - 2\,||x||\right) & ||x|| \geq 1/2. \end{cases}$$

If $||x|| = 1$ then $g(x) = F(x,0) = f(x)$. If $||x|| = 1/2$ then $g(x) = F(x,1) = c(x) = q$. Thus by the gluing lemma, g is continuous.

Now we prove (ii) $\Longrightarrow$ (iii). Let $g : B^{n+1} \longrightarrow Y$ be an extension of f and fix p. Define

$$\begin{aligned} F : \mathbb{S}^n \times I &\longrightarrow Y \\ (x,t) &\longmapsto g\left((1-t)x + tp\right). \end{aligned}$$

Then

$$\begin{aligned} F(x,0) &= g(x) = f(x), \quad \forall x \in \mathbb{S}^n \\ F(x,1) &= g(p) = f(p) = c(x). \end{aligned}$$

□

3 Homotopy functors

We will now develop all the necessary tools to define the homotopy functors π_n. First of all we will define π_0, a rather simple functor, and then we will make our way to π_1, the fundamental group. We begin by defining paths in a topological space.

Definition 3.1. A path u in a topological space X is a continuous map $u : I \longrightarrow X$, or more generally, from a space homeomorphic to an interval. A loop is a path such that $u(0) = u(1)$, in which case, u can be interpreted as a function $u : \mathbb{S}^1 \longrightarrow X$.

Usually, we will use the letters u, v and w for indicating paths, and we will parametrise a path with the letter s, since t is reserved for homotopies.

Definition 3.2. A topological space is path connected if $\forall x, y \in X$ there exists a path u such that $u(0) = x, u(1) = y$.

For a path u connecting two points $x, y \in X$ we will use the notation $u : x \longrightarrow y$. The following lemma is easily proved.

Lemma 3.3. Let X, Y be topological spaces.

(i) If X is path connected, then it is connected.

(ii) If X, Y are path connected, so is $X \times Y$.

(iii) If X is path connected and $f : X \longrightarrow Y$ is continuous, then $f(X)$ is path connected.

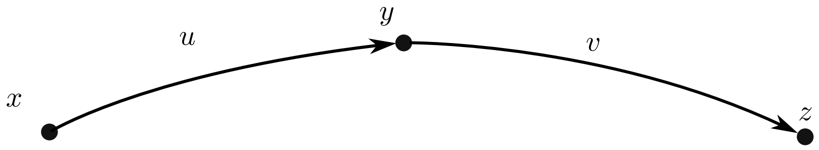
Here we prove the following equally easy result.

Lemma 3.4. Let X be a topological space. Define a relation $\sim$ on X by saying $x \sim y$ if there exists a path $u : x \longrightarrow y$. Then $\sim$ is a equivalence relation.

Proof. A constant path shows that $x \sim x$. For showing symmetry, it suffices to define $\bar{u}(s) := u(1-s)$ for $u : x \longrightarrow y$. For showing transitivity, if $u : x \longrightarrow y$ and $v : y \longrightarrow z$, then

$$(u * v)(s) = \begin{cases} u(2s) & s \in [0, 1/2] \\ v(2s-1) & s \in [1/2, 1] \end{cases}$$

is a path from x to z and it is continuous by the gluing lemma. ☐



Definition 3.5. The path components of X are the equivalence classes under $\sim$.

We now have everything we need to define the homotopy functor π_0.

Definition 3.6. We define $\pi_0 : \text{Top} \longrightarrow \text{Sets}$ by the following data.
On objects:

$$\pi_0(X) := \text{ the set of path components.}$$

On functions:

$$\begin{aligned} \pi_0 : \text{Hom}(X, Y) &\longrightarrow \quad \text{Hom}(\pi_0(X), \pi_0(Y)) \\ f : X \longrightarrow Y &\longmapsto \pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y). \end{aligned}$$

Given $x \in X$, π_0 sends the path component X' of X containing x to the path component of Y which contains $f(X')$.

To show π_0 is really a functor, we must show

$$\pi_0(id_X) = id_{\pi_0(X)},$$
$$\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f).$$

However, this is an easy exercise. What is interesting is how π_0 induces a functor on hTop.

Lemma 3.7. π_0 induces a functor $\text{hTop} \longrightarrow \text{Sets}$.

Proof. We need to show that if $f, g : X \longrightarrow Y$ are continuous functions such that $f \simeq g$, then $\pi_0(f) = \pi_0(g)$.

Let $F : X \times I \longrightarrow Y$ be a homotopy from f to g. Let X' be a path component of X. We want to show that the path component of Y containing $f(X')$ is the same as the one containing $g(X')$. Since $X' \times I$ is path connected and F is continuous, $F(X' \times I)$ is path connected, and there exists a path component Y' of Y such that $F(X' \times I) \subset Y'$. Since

$$f(X') = F(X' \times \{0\}) \subset F(X' \times I) \subset Y'$$
$$g(X') = F(X' \times \{1\}) \subset F(X' \times I) \subset Y'$$

we see that

$$\pi_0(f)(X') = Y'$$
$$\pi_0(g)(X') = Y'.$$

☐

We now define

$$\begin{array}{rccc} \pi_0 : & \text{hTop} & \longrightarrow & \text{Sets} \\ & X & \longmapsto & \pi_0(X) \\ & [f] : X \to Y & \longmapsto & \left\{ \begin{array}{rcl} \pi_0([f]) : \pi_0(X) & \longrightarrow & \pi_0(Y) \\ X'_x & \longmapsto & Y'_{f(x)} \end{array} \right\}. \end{array}$$

This is, regarding π_0 as a functor on the category hTop (where $[f]$ consists of all continuous functions $g : X \longrightarrow Y$ such that $g \simeq f$). Here X'_x denotes the path component containing $x \in X$ and $Y'_{f(x)}$ denotes the path component containing $f(x)$. More generally, we can state the following lemma, which indicates when a functor induces another functor in the quotient category.

Lemma 3.8. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $T : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor. Suppose $\sim$ is a congruence on $\mathcal{C}$ and T has the property that

$$f \sim g \Longrightarrow T(f) = T(g).$$

Then T induces a well defined functor

$$T : \mathcal{C}/\sim \longrightarrow \mathcal{D}.$$

Remark. If $T : \mathcal{C} \longrightarrow \mathcal{D}$ and f is an isomorphism in $\mathcal{C}$, then $T(f)$ is an isomorphism in $\mathcal{D}$. This fact is easy to check, since functors respect composition and identity maps.

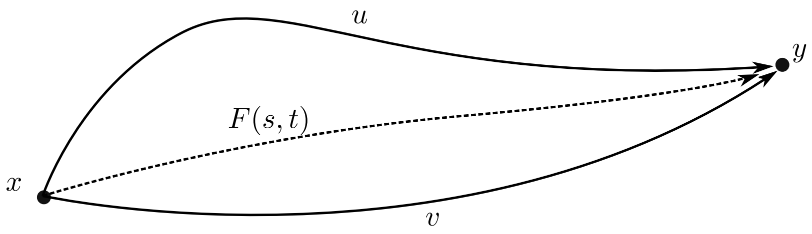
Corollary 3.9. Suppose X and Y are homotopy equivalent. Then X and Y have the same number of path components.

Proof. By the previous lemma, $\pi_0 :$ hTop $\longrightarrow$ Sets is a functor. An isomorphism $[f]$ in hTop is a homotopy equivalence. Since X, Y are homotopy equivalent, there exists $f : X \longrightarrow Y$ such that $[f] \in \text{Hom}_{\text{hTop}}(X, Y)$ is an isomorphism. This means that $\pi_0([f])$ is an isomorphism in Sets, which means that it is a bijection: $\pi_0(X)$ is bijective with $\pi_0(Y)$ and thus they have the same cardinality. $\square$

Since the category of sets is not really interesting due to its structure not being very rich, this is the end of the story for π_0, it will not get any more interesting that this. Now, we will define π_1, which lands in the category of groups, one that is much richer. The idea behind π_1 will be the concatenation of paths.

Definition 3.10. Suppose $u : x \longrightarrow y$. We write $[u]$ for the relative homotopy class $[u]$ rel ∂I.

It makes sense to consider this class of homotopy of paths, since we want the starting and ending points of the path to remain constant throughout the homotopy. Since X is path connected and I is contractible, every path $u : I \longrightarrow X$ is homotopic, and if we tried to construct a group form homotopy classes of paths, this group would be trivial. If we unravel the definition above, this is equivalent to saying $u \simeq v$ rel ∂I if and only if there exists a homotopy $F : I_s \times I_t \longrightarrow Y$ such that $F(s, t) = u(s) = v(s)$ for $s = 0, 1$ and for all $t \in I$.



Before we continue, let us properly define the concatenation of paths.

Definition 3.11. Let u and v be paths in X with $u(1) = v(0)$. Then we define

$$(u * v)(s) = \begin{cases} u(2s) & s \in [0, 1/2] \\ v(2s-1) & s \in [1/2, 1]. \end{cases}$$

Remark. Note that the ordering here is the opposite to composition. $u * v$ means "first do u, then do v", and $g \circ f$ means "first do f, then do g".

Proposition 3.12. Suppose u_0, u_1, v_0, v_1 are paths such that

$$\begin{aligned} u_0, u_1 &: x \longrightarrow y \\ v_0, v_1 &: y \longrightarrow z \end{aligned}$$

and $[u_0] = [u_1]$, $[v_0] = [v_1]$. Then $[u_0 * v_0] = [u_1 * v_1]$

Proof. Let $U : I \times I \longrightarrow X$ such that

$$\begin{aligned} U(s,0) &= u_0(s) \\ U(s,1) &= u_1(s) \\ U(0,t) &= U(0,0) \\ U(1,t) &= U(1,0). \end{aligned}$$

Let V, W be homotopies

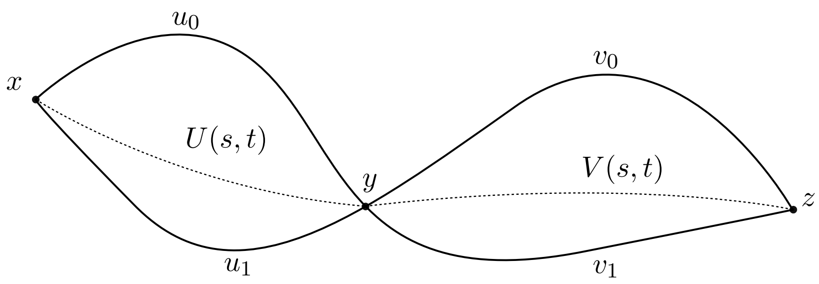
$$\begin{aligned} V &: v_0 \simeq v_1 \quad \text{rel } \partial I \\ W &: u_0 * v_0 \simeq u_1 * v_1 \end{aligned}$$

and define W as

$$W(s,t) = \begin{cases} U(2s,t) & 0 \leq s \leq 1/2 \\ V(2s-1,t) & 1/2 \leq s \leq 1. \end{cases}$$

Then W is continuous by the gluing lemma. Also,

$$\begin{aligned} W(s,0) &= u_0 * v_0(s) \\ W(s,1) &= u_1 * v_1(s) \\ W(0,t) &= U(0,t) = U(0,0) = W(0,0) \\ W(1,t) &= W(1,0). \end{aligned}$$



□

3.1 Fundamental grupoid

We take a look at a different category from all the ones we have seen until now.

Definition 3.13. Let X be a topological space. We define a category $\Pi(X)$ called the fundamental grupoid of X as follows.

(i) **Objects:** $\text{obj}(\Pi(X)) = X$.

(ii) **Morphisms:** let $x, y \in X = \text{obj}(\Pi(X))$. We define

$$\text{Hom}_{\Pi(X)}(x,y) = \Pi(x,y) = \{[u] : \ u \text{ is a path from } x \text{ to } y.\}$$

(iii) **Composition:**

$$\begin{array}{rcl} \circ : \Pi(x,y) \times \Pi(y,z) & \longrightarrow & \Pi(x,z) \\ ([u],[v]) & \longmapsto & [u * v] \end{array} .$$

Indeed, saying $\Pi(X)$ is a category is true, but we have to actually give a proof. Let us do so.

Theorem 3.14. $\Pi(X)$ is a category.

Proof. We need to prove that composition $\circ$ is associative where defined, and that the sets $\Pi(x,x)$ have an identity element. Let us deal with the latter first.

Denote

$$\begin{aligned} e_x : I &\longrightarrow X \\ s &\longmapsto x. \end{aligned}$$

We claim that $[e_x] \in \Pi(x,x)$ is an identity element under the above composition. To prove this, we have to check that for $[u] \in \Pi(x,y), [v] \in \Pi(z,x)$,

$$\begin{aligned} [u] \circ [e_x] &= [e_x * u] = [u] \\ [e_x] \circ [v] &= [e_x * v] = [v]. \end{aligned}$$

Define

$$l_t(s) = \frac{s - \frac{1}{2}(1-t)}{1 - \frac{1}{2}(1-t)}.$$

Then l_t maps L_t in figure 3.1 to $[0,1]$. Now consider the map $U : I \times I \longrightarrow X$ given by

$$U(s,t) := \begin{cases} x \quad 2s & \leq 1-t \\ u(l_t(s)) \quad 2s & \geq 1-t. \end{cases}$$

The gluing lemma shows that U is continuous, and by construction $U : e_x \simeq u$ rel ∂I.

Now let us check the composition is associative. Let

$$\begin{aligned} [u] &\in \Pi(x,y) \\ [v] &\in \Pi(y,z) \\ [w] &\in \Pi(z,a) \end{aligned}$$

We must show

$$\begin{gathered} [w] \circ ([v] \circ [u]) \\ ([w] \circ [v]) \circ [u] \end{gathered}$$

are both in $\Pi(x,a)$, that is

$$[(u * v) * w] = [u * (v * w)].$$

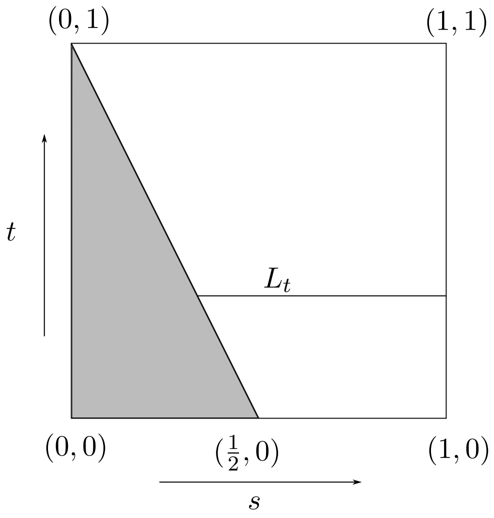


Figure 3.1: proving $e_x * u \simeq u$ rel ∂I.

For this, let L_t, M_t and N_t be as in figure 3.2. Now let l_t, m_t and n_t be the parametrisations that map L_t, M_t and N_t onto $[0, 1]$ respectively. The desired homotopy is obtained by setting $U(s, t) = u(l_t(s))$ on the left–hand region, $U(s, t) = v(m_t(s))$ on the middle region and finally $U(s, t) = w(n_t(s))$ on the right–hand region. The gluing lemma shows that U is continuous, and by construction we have $U : (u * v) * w \simeq u * (v * w)$ rel ∂I.

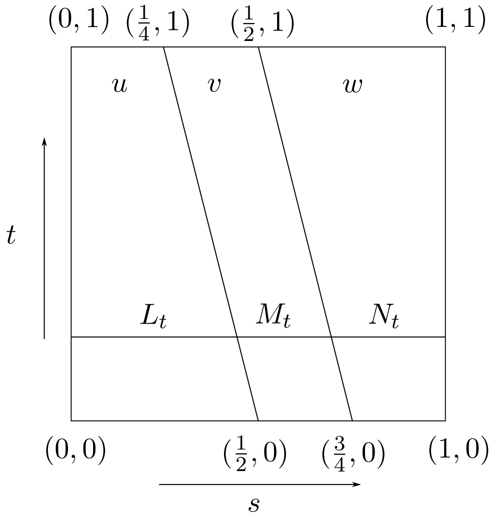


Figure 3.2: proving associativity.

Thus the fundamental grupoid is a category. ☐

In fact, the fundamental grupoid is a category with one special property.

Definition 3.15. A grupoid category $\mathcal{C}$ is a category that fulfills the following requirements.

(i) the category is "small" in the sense that obj($\mathcal{C}$) form a set (and not just a class).

(ii) Every morphism is an isomorphism. If $A, B \in$ obj($\mathcal{C}$) and $f \in$ Hom(A, B), then exists $g \in \operatorname{Hom}(B, A)$ such that

$$f \circ g = id_B$$
$$g \circ f = id_A.$$

Remark. If $\mathcal{C}$ is a grupoid category and $A \in \operatorname{obj} \mathcal{C}$, then $\operatorname{Hom}_{\mathcal{C}}(A, A)$ is a group.

Let us take a look at an example. Let G be a group and define a category $\mathcal{C}$ with exactly one object $*$. Define $\operatorname{Hom}(*, *) = G$ and composition is multiplication in G. This is our first example of a category where composition cannot be thought of as "mapping from one object to another". In $\mathcal{C}$, for $g \in G$ to be an isomorphism it is required that there exists $h \in G$ such that g and h are inverses with respect to group multiplication, but this happens for every $g \in G$.

Another example: let X be a topological space. Then $\Pi(X)$ is a grupoid category. To see this, one first checks that $\operatorname{obj} \Pi(X) = X$, which is a set. The second part consists of proving that every morphism is an isomorphism. It is easy to see that for every path $[u] \in \Pi(x, y)$, its backwards path $[\bar{u}]$ is in $\Pi(y, x)$, and

$$[\bar{u}] \circ [u] = [u * \bar{u}] = [e_{u(0)}]$$
$$[u] \circ [\bar{u}] = [\bar{u} * u] = [e_{u(1)}].$$

3.2 The fundamental group

Definition 3.16. Let X be a topological space and fix $p \in X$. The fundamental group of X at p is

$$\pi_1(X, p) = \operatorname{Hom}_{\Pi(X)}(p, p) = \{\text{path classes of loops based at } p\}.$$

Remark. π_0, π_1 are often easy to compute. However, higher π_i's are really difficult to compute. In fact, it is an open problem to find $\pi_i(\mathbb{S}^n)$ for all $i, n \in \mathbb{N}$.

For example, $\pi_1(\mathbb{S}^1, \cdot) = \mathbb{Z}$ and $\pi_1(B^n, \cdot) = 0$. This means that both fundamental groups are the same for every point in each of the two sets. We will later give a result on that note.

Definition 3.17. The category Top^2 is the category given by the following data.

(i) Objects are pairs (X, X') where X is a topological space and $X' \subset X$.

(ii) Morphisms are pairs f, f' such that $f : X \longrightarrow Y'$, $f' : X' \longrightarrow Y'$ and the diagram below commutes.

$$\operatorname{Hom}_{\mathrm{Top}^2}\left((X, X'), (Y, Y')\right) = \left\{f : X \longrightarrow Y,\ f(X') \subset Y'\right\}.$$

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X' \arrow[u, hook, "i"] \arrow[r, "f'"] & Y' \arrow[u, hook, "i"]
\end{tikzcd}$$

Lemma 3.18. Let X, Y, Z be topological spaces. Let $X' \subset X$ and $Y' \subset Y$. Suppose f_0, f_1 are continuous maps such that $f_0|_{X'} = f_1|_{X'}$ and $f_0(X') \subset Y'$. Suppose g_0, g_1 are continuous maps $Y \longrightarrow Z$ such that $g_0|_{Y'} = g_1|_{Y'}$. Suppose $f_0 \simeq f_1$ rel X' and $g_0 \simeq g_1$ rel Y'. Then $g_0 \circ f_0 \simeq g_1 \circ f_1$ rel X'.

Remark. We can think of Top as sitting inside Top^2 via

$$X \longrightarrow (X, \emptyset).$$

Now we will take a look at the special case in which $X' = \{p\} = p \, (\in X)$.

Definition 3.19. Top$_*$ is the pointed topological category, defined as follows.

(i) **Objects:** (X, p)

(ii) **Morphisms:** continuous maps

$$f : (X, p) \longrightarrow (Y, q)$$

where $f : X \longrightarrow Y$ is a continuous map such that $f(p) = q$. Such maps are called pointed maps.

Now we see that in the category of pointed topological spaces Top$_*$, π_1 is a functor.

Proposition 3.20. The fundamental group is a functor

$$\begin{aligned} \pi_1 : \text{Top}_* &\longrightarrow \text{Groups} \\ (X, p) &\longmapsto \pi_1(X, p). \end{aligned}$$

On morphisms, $f : (X, p) \longrightarrow (Y, q)$,

$$\begin{aligned} \pi_1(f) : \pi_1(X, p) &\longrightarrow \pi_1(Y, q) \\ [u] &\longmapsto [f \circ u]. \end{aligned}$$

Lemma 3.21. Suppose $f, g : X \longrightarrow Y$ such that $f(p) = g(p) = q$ and

$$f \simeq g \text{ rel } p.$$

Then the induced maps

$$\pi_1(f), \pi_1(g) : \pi_1(X, p) \longrightarrow \pi_1(Y, q)$$

coincide.

Proof. Apply lemma 3.18 with $X = I, Y = X, Z = Y, X' = \partial I, Y' = \{p\}$ and $f_0 = u_0, f_1 = u_1, g_0 = f, g_1 = g$. ☐

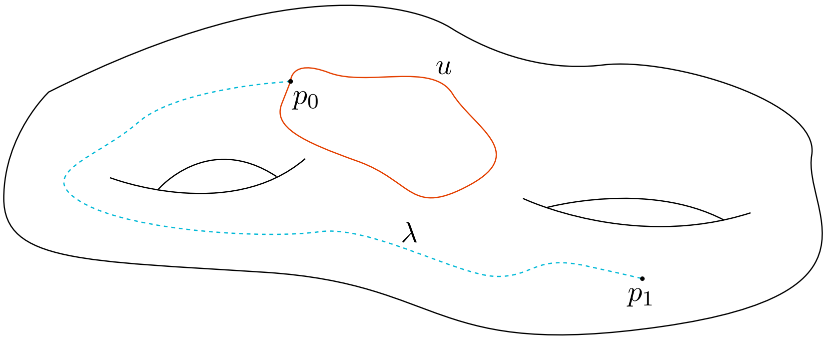
hTop was constructed from Top via the congruence of homotopy. Similarly, hTop$_*$ is constructed from Top$_*$ via the congruence of homotopy relative to a point. Lemma 3.21 proves that π_1 will be well defined over hTop$_*$.

Corollary 3.22. The fundamental group is a functor

$$\mathrm{hTop}_* \longrightarrow \mathrm{Groups}\,.$$

Corollary 3.23. If (X,p) and (Y,q) are isomorphic objects in hTop_*, then $\pi_1(X,p) \simeq \pi_1(Y,f(p))$.

To prove this one simply unravels the definition of isomorphism in hTop_*. A natural question to ask ourselves now is how does $\pi_1(X,p)$ depends on p. Let us suppose X is path connected.



Since the elements of the fundamental grupoid are classes of relative homotopy, we can factor concatenation in the following way.

$$[\bar{\lambda} * u * \lambda] = [\bar{\lambda}] * [u] * [\lambda].$$

This only makes sense on $\Pi(X)$ since u and v have the same values at ∂I for all $v \in [u]$. Now we can think of the maps

$$\begin{array}{ccc} \pi_1(X,p_1) & \longrightarrow & \pi_1(X,p_0) \\ {[v]} & \longmapsto & [\lambda * v * \bar{\lambda}] \end{array}$$

and

$$\begin{array}{ccc} \pi_1(X,p_0) & \longrightarrow & \pi_1(X,p_1) \\ {[u]} & \longmapsto & [\bar{\lambda} * u * \lambda]. \end{array}$$

The composition of both of them gives

$$[u] \mapsto [\bar{\lambda} * u * \lambda] \mapsto [\lambda * \bar{\lambda} * y * \lambda * \bar{\lambda}] = [u]$$

and similarly for $[v]$. This indicates that both functions are isomorphisms of groups. Therefore, for path connected spaces, the fundamental groups are independent of the basepoint up to isomorphisms.

Proposition 3.24. Suppose we have $f_0, f_1 : X \longrightarrow Y$ which are homotopic, but not necessarily homotopic relative to p. Set $q_0 = f_0(p)$ and $q_1 = f_1(p)$. Then the following diagram commutes

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "\pi_1(f_0)"] \arrow[dr, "\pi_1(f_1)"'] & \pi_1(Y,q_0) \arrow[d, "\varphi_\lambda : {[u]} \longrightarrow {[\bar{\lambda} * u * \lambda]}"] \\ & \pi_1(Y,q_1) \end{tikzcd}$$

Proof. Let $F : f_0 \simeq f_1$ and let λ denote the path in Y given by $\lambda(t) = F(p,t)$. Choose $[u] \in \pi_1(X,p)$ and consider

$$\begin{aligned} V : I \times I &\longrightarrow \qquad Y \\ (s,t) &\longmapsto \begin{cases} F\left(u(2(1-t)s), 2st\right) & s \leq 1/2 \\ F\left(u(1+2t(s-1)), t+(2s-1)(1-t)\right) & s \geq 1/2. \end{cases} \end{aligned}$$

For $s = 1/2$,

$$\begin{aligned} F\left(u(2(1-t)\frac{1}{2}), 2\frac{1}{2}t\right) &= u((1-t),t) \\ F\left(u\left(1+2t\left(\frac{1}{2}-1\right)\right), t+\left(2\frac{1}{2}-1\right)(1-t)\right) &= F(u(1-t),t) \end{aligned}$$

F is continuous by the gluing lemma. Now,

$$V(s,0) = \begin{cases} F(u(2s),0) = f_0 \circ u(2s) & s \leq 1/2 \\ F(u(1),2s-1) = \lambda(2s-1) & s \geq 1/2 \end{cases}$$

and

$$V(s,1) = \begin{cases} F(u(0),2s) = \lambda(2s) & s \leq 1/2 \\ F(u(2s-1),1) = f_1(u(2s-1)) & s \geq 1/2 \end{cases}$$

Thus, we have

$$\begin{aligned} V(s,0) &= (f_0 \circ u) * \lambda(s) \\ V(s,1) &= \lambda * (f_1 \circ u)(s) \\ V(0,t) &= F(u(0),0) = f_0(p) = q_0 \\ V(1,t) &= F(u(1),1) = f_1(p) = q_1. \end{aligned}$$

Putting this all together, V is a homotopy from $(f_0 \circ u) * \lambda$ to $\lambda * (f_1 \circ u)$ relative to ∂I. This means that

$$\begin{aligned} [f_0 \circ u] * [\lambda] &= [\lambda] * [f_1 \circ u] \\ [\bar{\lambda} * (f_0 \circ u) * \lambda] &= [f_1 \circ u]. \end{aligned}$$

Finally, $\varphi_\lambda \circ \pi_1(f_0)[u] = \pi_1(f_1)[u]$. ☐

Corollary 3.25. Assume $q_0 = q_1$. Then the maps $\pi_1(f_0)$ and $\pi_1(f_1)$ are conjugate group homomorphisms.

Proof. In this case λ is a loop, $F(p,t)$ has the property that $F(p,0) = F(p,1)$, so it is homotopic rel p to a constant map. Then

$$\pi_1(f_1)[u] = [\bar{\lambda}] * \pi_1(f_0)[u] * [\lambda].$$

☐

Proposition 3.26. If $f : X \longrightarrow Y$ is a homotopy equivalence, then for any $p \in X$ the map $\pi_1(f) : \pi_1(X,p) \longrightarrow \pi_1(Y, f(p))$ is a group isomorphism.

Proof. By assumption there exists $g : Y \longrightarrow X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Let $F : g \circ f \simeq id_X$. As before, let $\lambda(t) = F(p,t)$. Then we have the following commutative diagram.

$$\begin{tikzcd}
& \pi_1(Y, f(p)) \arrow[dr, "\pi_1(g)"] & \\
\pi_1(X,p) \arrow[ur, "\pi_1(f)"] \arrow[rr, "\pi_1(g\circ f)"] \arrow[dr, "id"] & & \pi_1(X, g\circ f(p)) \arrow[dl, "\varphi_\lambda"] \\
& \pi_1(X,p) &
\end{tikzcd}$$

The top half commutes as π_1 is a functor. The bottom half commutes by the previous proposition. Thus $\pi_1(f)$ is injective and $\pi_1(g)$ is surjective. As a consequence, $\pi_1(f)$ is surjective and $\pi_q(g)$ is injective. □

Remark.

(i) We have seen that for p, q in the same path component of X,

$$\pi_1(X,p) \cong \pi_1(X,q)$$

via the appropriate φ_λ. However, the isomorphism φ_λ depends on $[\lambda]$. We can therefore write $\pi_1(X)$ without reference to p, but the group is only defined up to isomorphism.

(ii) If $X = \bigcup X_i$ is the decomposition into path components of X, if $x \in X_i$, then $\pi(X,p) \cong \pi(X_i,p)$.

Corollary 3.27. $\pi_1(B^n) = 0$.

Now we will show an example of a path connected space whose fundamental group is not trivial, the unit circle $\mathbb{S}^1$, which we will regard as a subset of $\mathbb{C}$. Throughout the rest of this section,

$$\mathbb{S}^1 = \{z \in \mathbb{C} : \ |z| = 1\}.$$

Let us denote

$$\begin{aligned} \exp : \mathbb{R} &\longrightarrow \mathbb{S}^1 \subset \mathbb{C} \\ s &\longmapsto e^{2\pi i s}. \end{aligned}$$

We will be interested in the case where the following diagram commutes,

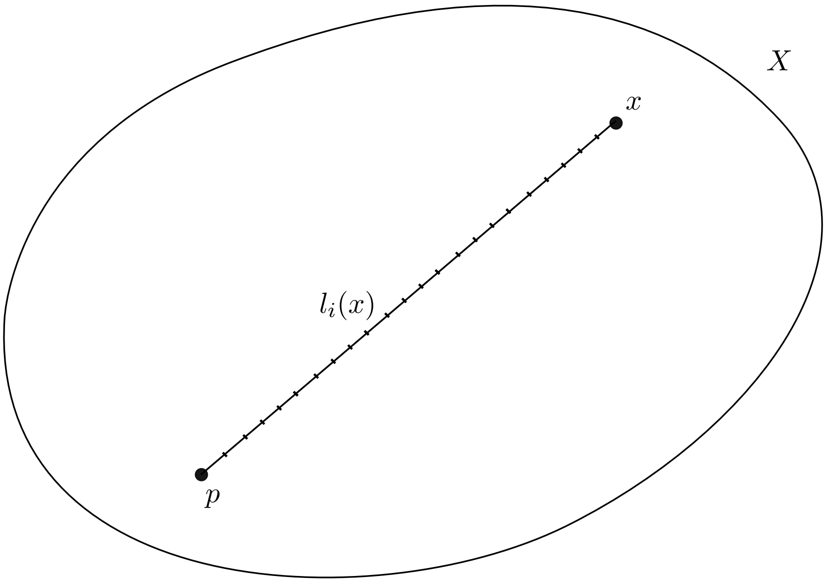
$$\begin{tikzcd}
(X,p) \arrow[r, "f"] \arrow[dr, "\tilde{f}"'] & (\mathbb{S}^1, 1) \\
& (\mathbb{R}, m) \arrow[u, "\exp"]
\end{tikzcd}$$

where $m \in \mathbb{Z}$. We will also make use of the fact that $\mathbb{S}^1$ is itself a group. Given $z = e^{2\pi i s}$ and $w = e^{2\pi i t}$ in $\mathbb{S}^1$, their product $zw = e^{2\pi i(s+t)} \in \mathbb{S}^1$, and the inverse of $z = e^{2\pi i s}$ is just $z^{-1} = e^{-2\pi i s}$. Also, we say z and w are antipodal if $zw^{-1} = -1$, which of course has the geometric interpretation that z and w are opposite with respect to the origin.

Proposition 3.28. Let X be a compact convex subset of $\mathbb{R}^n$. Suppose $f : (X, p) \longrightarrow (\mathbb{S}^1, 1)$. Fix $m \in \mathbb{Z}$. Then there exists a unique $\tilde{f} : (X, p) \longrightarrow (\mathbb{R}, m)$ such that the above diagram commutes.

This is, $f = \exp \circ \tilde{f}$. In this context $\tilde{f}$ is called a lift of f. In particular, one can think of $\mathbb{S}^1$ as $\mathbb{R}/\mathbb{Z}$ and thus $\tilde{f}$ is "lifting" f from the quotient space to $\mathbb{R}$. Note also that the requirement $m \in \mathbb{Z}$ is forced: we have $\tilde{f}(p) = s \in \mathbb{R}$ and $f(p) = 1$, so it must be $\exp(s) = 1$, thus making $s \in \mathbb{Z}$.

Proof. We will define $\tilde{f}$. Compactness of X implies uniform continuity of f. Thus there exists $\varepsilon > 0$ such that if $x, y \in X$ with $|x - y| < \varepsilon$, then $f(x)$ and $f(y)$ are not antipodal. Since X is bounded there exists an integer N such that $|x - y| < N\varepsilon$ for all $x, y \in X$. Now for each $x \in X$, subdivide the line segments with endpoints p an x into N intervals of equal length (which are all contained in X since it is convex). Call the endpoints of these intervals $p = l_0(x), ..., l_N(x) = x$.



It is easy to check that the functions $l_1, ..., l_n$ are continuous. Define

$$\begin{aligned} g_i : X &\longrightarrow \mathbb{S}^1 \setminus \{-1\} \\ x &\longmapsto f(l_i(x))^{-1} \cdot f(l_{i+1}(x)). \end{aligned}$$

Since $f(l_i(x))$ and $f(l_{i+1}(x))$ are never antipodal, $g_i(x)$ is never -1 and it is continuous. Note also that $g_i(p) = 1$ for all i. Now, because $l_N(x) = x$, we can write

$$\begin{aligned} f(x) = f(p)f(p)^{-1}f(l_N(x)) &= f(p)f(p)^{-1}f(l_1(x))f(l_1(x))^{-1} \dots f(l_N(x)) = \\ &= f(p)f(p)^{-1}g_1(x) \dots g_{N-1}(x) = f(p)^{-1}g_1(x) \dots g_{N-1}(x). \end{aligned}$$

If we restrict exp to $(-1/2, 1/2)$ then it is a homeomorphism onto its image $\mathbb{S}^1 \setminus \{-1\}$. Let $\Lambda : \mathbb{S}^1 \setminus \{-1\} \longrightarrow (-1/2, 1/2)$ denote its inverse, $\Lambda = \frac{1}{2\pi i} \log$. Since g_i never attains the value -1, the composition $\Lambda \circ g_i : X \longrightarrow (-1/2, 1/2)$ is well defined.

Define

$$\tilde{f}(x) = m + \sum_{i=1}^{N-1} \Lambda \circ g_i(x).$$

Then $\tilde{f}(p) = m$ and $\exp \circ \tilde{f}(x) = f(x)$.

Let us now prove uniqueness. Suppose $\tilde{g}$ was another map such that $\tilde{g}(p) = m$ and $\exp \circ \tilde{g} = f$. Consider $\tilde{h} = \tilde{f} - \tilde{g}$. Then $\tilde{h}(p) = 0$, $\exp \circ \tilde{h} \equiv 1$. Thus $\tilde{h}$ is a continuous function with values in $\mathbb{Z}$ and $\tilde{h}(0) = 0$. Since $\mathbb{Z}$ is discrete, it must happen that $\tilde{h} \equiv 0$. ☐

Corollary 3.29.

(i) Suppose $u : I \longrightarrow \mathbb{S}^1$ with $u(0) = u(1) = 1$. Then there exists a unique lift $\tilde{u} : I \longrightarrow \mathbb{R}$ such that $\tilde{u}(0) = 0$.

(ii) Moreover, if $v : I \longrightarrow \mathbb{S}^1$ is another loop such that $v(0) = v(1) = 1$ and $u \simeq v$ rel ∂I, then $\tilde{u} \simeq \tilde{v}$ rel ∂I. In particular $\tilde{u}(1) = \tilde{v}(1)$.

Proof. The first part follows from the proposition taking $X = I$. For the second part, let $U : I \times I \longrightarrow \mathbb{S}^1$ be a homotopy $u \simeq v$ rel ∂I. The proposition gives a unique $\tilde{U} : I \times I \longrightarrow \mathbb{R}$ such that $\tilde{U}(0,0) = 0$ and $\exp \circ \tilde{U} = U$. We claim that $\tilde{U} : \tilde{u} \simeq \tilde{v}$ rel ∂I.

First note $\tilde{U}(s,0)$ is a lift of u with $\tilde{U}(0,0) = 0$. Thus $\tilde{u} = \tilde{U}(\cdot,0)$ by uniqueness. Similarly, $\tilde{v} = \tilde{U}(\cdot,1)$. It remains to prove that this homotopy is relative to ∂I.

Consider $\tilde{U}(0,t)$. Then $\exp(\tilde{U}(0,t)) = U(0,t) = u(t) = 1$ since $U : u \simeq v$ rel ∂I and $v(0) = 1$. Thus $\tilde{U}(0,t) = 0$. Similarly, we have $\exp \circ \tilde{U}(1,t) = U(1,t) = v(1) = 1$, and $\tilde{U}(1,t)$ is a constant function by uniqueness. This constant is equal to $\tilde{u}(0) = \tilde{U}(1,0) = \tilde{U}(1,1) = \tilde{v}(1)$. ☐

The second statement in the corollary tells us that homotopic loops are those that turn around the origin the same number of times in the same direction.

Definition 3.30. Given $u : I \longrightarrow \mathbb{S}^1$, $u(0) = u(1) = 1$, we define $\deg(u) := \tilde{u}(1)$, where $\tilde{u}$ is the unique lift with $\tilde{u}(0) = 0$.

Then

$$\begin{aligned} \deg : \pi_1(\mathbb{S}^1, 1) &\longrightarrow \mathbb{Z} \\ [u] &\longmapsto \deg(u) \end{aligned}$$

is well defined by the last statement in the corollary. Note that since $u(1) = u(0) = 1$, we have $\tilde{u}(1) \in \mathbb{Z}$ and $\deg(u)$ indicates the number of times u revolves around the origin.

Proposition 3.31. $\deg : \pi_1(\mathbb{S}^1, 1) \longrightarrow \mathbb{Z}$ is a group isomorphism.

$$\deg([u] * [v]) = \deg([u]) + \deg([v]).$$

Proof. It is trivial that deg is surjective since $s \mapsto ms$ has degree m. It can also be seen as the map $\mathbb{S}^1 \ni z \mapsto z^m$. To prove injectivity, we note that if $\deg[u] = 0$ then $\tilde{u}$ is a loop in $\mathbb{R}$ based at 0. Since $\pi_1(\mathbb{R}, 0) = \{1\}$, by the last statement of the corollary, and since $\pi_1(\exp) : \pi_1(\mathbb{R}, 0) \longrightarrow$

$\pi_1(\mathbb{S}, 1)$ is a group homomorphism (this is because exp is a group homomorphism and π_1 is a functor), it follows that $[u] = \pi_1(\exp)[\tilde{u}]$ and thus $[u] = 1$. All that is left to show is that deg is in fact a homomorphism.

Now, if we have u, v, we get $\tilde{u}, \tilde{v}$ and we want to find what $\widetilde{u * v}$ is. Let $m = \deg(u)$ so that $\tilde{u}(1) = m$. Let $\tilde{w} := m + \tilde{v}$. Then $\tilde{u}(1) = \tilde{w}(0)$, so $\tilde{u} * \tilde{w}$ is well defined. Now we note that $\exp(\tilde{u} * \tilde{w}) = u * v$ and $(\tilde{u} * \tilde{w})(0) = 0$. Thus $\tilde{u} * \tilde{w}$ is a lift of $u * v$ which starts at 0. Thus

$$\begin{aligned}\deg(u * v) &= (\tilde{u} * \tilde{w})(1) = m + \tilde{v}(1) = \\ &= \deg(u) + \deg(v).\end{aligned}$$

□

4 Pushouts

Definition 4.1. Suppose $\mathcal{C}$ is a category and A, B, C are three objects in $\mathcal{C}$. Suppose $f : A \longrightarrow B$ and $g : A \longrightarrow C$ are two morphisms. A diagram in $\mathcal{C}$ is a picture of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C &
\end{tikzcd}$$

A solution of this diagram is an object D and two morphisms $h \in \text{Hom}(B, D)$, $k \in \text{Hom}(C, D)$ such that the diagram below commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "k"] & D
\end{tikzcd}$$

Further along the course we will define a more general notion of diagram which allows for pictures of different shapes. Solutions are not necessarily unique. For example, if $\mathcal{C} = \text{Sets}$ and $A = \{*\}$. Then given $f : A \longrightarrow B$ and $g : A \longrightarrow C$, one option is $D = B \times C$. However, we will be interested in finding the "most efficient" solution.

Definition 4.2. A pushout of a diagram

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \\
B_2 &
\end{tikzcd}$$

is a solution

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"] \\
B_2 \arrow[r, "g_2"] & C
\end{tikzcd}$$

with the following property. If

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "h_1"] \\
B_2 \arrow[r, "h_2"] & D
\end{tikzcd}$$

is any solution, then there exists a unique map $k : C \longrightarrow D$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"] \arrow[rdd, bend left, "h_1"] & \\
B_2 \arrow[r, "g_2"] \arrow[rrd, bend right, "h_2"'] & C \arrow[rd, "k"] & \\
& & D
\end{tikzcd}$$

Lemma 4.3. If a pushout exists, then it is unique up to isomorphism.

Proof. Suppose (C, g_1, g_2) and (D, h_1, h_2) are two pushouts. Then there exist unique $k : C \longrightarrow D$ and $l : D \longrightarrow C$ such that the diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"] \arrow[rdd, bend left, "h_1"] & \\
B_2 \arrow[r, "g_2"] \arrow[rrd, bend right, "h_2"'] & C \arrow[rd, shift left, "k"] & \\
& & D \arrow[lu, shift left, "l"]
\end{tikzcd}$$

We have $k \circ g_1 = h_1$ and $k \circ g_2 = h_2$. Also, $l \circ h_1 = g_1$ and $l \circ h_2 = g_2$. We note that

$$l \circ k \circ g_1 = l \circ h_1 = g_1$$
$$k \circ l \circ h_1 = k \circ g_1 = h_1.$$

Now, since C is a pushout and a solution as well, there must exist a unique map $C \longrightarrow C$ such that the diagram above with C instead of D commutes. However, both id_C and $l \circ k$ make it so the diagram commutes. By uniqueness, $l \circ k = id_C$. By the same argument with D, $k \circ l = id_D$. Thus $k : C \cong D$ in $\mathcal{C}$. □

For example, in the category of commutative rings, $f_1 : A \longrightarrow B_1$, $f_2 :\longrightarrow B_2$, a pushout is $B_1 \otimes_A B_2$, the tensor product of B_1 and B_2. We thus have that the tensor product is unique.

Definition 4.4. A mathematical object satisfies a universal property if it can be defined via a pushout of some diagram.

Lemma 4.5. If a mathematical object exists and can be defined via a universal property then it is unique up to isomorphism.

In Sets, for

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \\
B_2 &
\end{tikzcd}$$

the pushout is $B_1 \bigsqcup B_2/\sim$ where $\sim$ is the smallest equivalence relation such that $f_1(a) \sim f_2(a)$ for all $a \in A$.

In hTop such a space is called an adjunction space

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, "f_2"'] & Y_1 \\
Y_2 &
\end{tikzcd}$$

It is denoted by $Y_1 \bigcup_X Y_2$ and we will study it further in the course.

Now take $\mathcal{C} =$ Groups. What is a pushout?

Definition 4.6. Let G and H be groups. A word in G and H of length n is an expression

$$s_1 ... s_n, \quad s_i \in G \text{ or } H.$$

A word can be reduced as follows. If s_i and s_{i+1} both belong to G, we can regard $s_i s_{i+1}$ as $\tilde{s}_i$, so that the word reduces to

$$s_1 ... s_{i-1} \tilde{s}_i s_{i+2} ... s_n.$$

If s_j and s_{j+1} both belong to H and, moreover, $s_j = (s_{j+1})^{-1}$, then the word is reduced as

$$s_1 ... s_{j-1} s_{j+2} ... s_n.$$

A reduced word is a word which cannot be further reduced.

Definition 4.7. The free product of G and H, written $G * H$, is the group of reduced words.

Remark. The free product $G * H$ can be defined via a universal property.

Proposition 4.8. Let G, H_1, H_2 be groups and suppose $\phi_1 : G \longrightarrow H_1$, $\phi_2 : G \longrightarrow H_2$ are group homomorphisms.

$$\begin{tikzcd}
G \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & H_1 \\
H_2 &
\end{tikzcd}$$

Let N be the smallest normal subgroup of $H_1 * H_2$ containing the subgroup generated by elements of the form $\phi_1(g^{-1})\phi_2(g)$ for $g \in G$. Then the quotient group $H_1 * H_2/N$ is a pushout. It is called the "free product with amalgamation" and usually denoted by $H_1 *_G H_2$.

Proposition 4.8 above makes sure that pushouts always exist in the category of groups. Our objective right now is to prove proposition 4.8 in order to obtain the necessary tools for proving our first difficult result, the Seifert–van Kampen theorem. But before we start proving the proposition, we have to take a look at the following definition.

Definition 4.9. Let $\mathcal{C}$ be a category and fix $A_1, A_2 \in \operatorname{obj}\mathcal{C}$. A coproduct in $\mathcal{C}$ is a triple (B, f_1, f_2) where B is another object, $f_i \in \operatorname{Hom}(A, B)$ and the following universal property holds.

If C is another object and $g_1 \in \operatorname{Hom}(A_1, C)$ and $g_2 \in \operatorname{Hom}(A_2, C)$, then there exists a unique morphism $h : B \longrightarrow C$ such that the following diagram commutes.

$$\begin{tikzcd}
 & B \arrow[dd, "h"] & \\
A_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & & A_2 \arrow[ul, "f_2"'] \arrow[dl, "g_2"] \\
 & C &
\end{tikzcd}$$

Remark.

(i) If a coproduct exists, it is unique up to isomorphism.

(ii) In the category Groups, the free product is the coproduct.

Now we have the necessary tools to prove proposition 4.8.

Proof of proposition 4.8. Let N be the normal subgroup of the free product $H_1 * H_2$ generated by elements of the form $\phi_1(g^{-1})\phi_2(g)$, and let K be the quotient group. Define for $i = 1, 2$

$$\begin{aligned} \psi_i : H_i &\longrightarrow K \\ h &\longmapsto hN. \end{aligned}$$

Let us check that (K, ψ_1, ψ_2) is a solution.

$$\begin{tikzcd}
G \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & H_1 \arrow[d, "\psi_1"] \\
H_2 \arrow[r, "\psi_2"] & K
\end{tikzcd}$$

We must show that for all $g \in G$, as cosets in K, we have

$$\phi_1(g)N = \phi_2(g)N$$

or equivalently

$$\phi_1(g)^{-1}\phi_2(g)N = N.$$

Since ϕ_1 is a homomorphism, $\phi_1(g^{-1}) = \phi_1(g)^{-1}$, and since $\phi_1(g)^{-1}\phi_2(g) \in N$ by choice of N, (K, ψ_1, ψ_2) is a solution. Now let

$$\begin{tikzcd}
G \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & H_1 \arrow[d, "\theta_1"] \\
H_2 \arrow[r, "\theta_2"] & L
\end{tikzcd}$$

be another solution. We need to produce a unique homomorphism $\tau : K \longrightarrow L$ such that the following commutes.

$$\begin{tikzcd}
G \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & H_1 \arrow[d, "\psi_1"] \arrow[ddr, bend left, "\theta_1"] & \\
H_2 \arrow[r, "\psi_2"] \arrow[drr, bend right, "\theta_2"'] & K \arrow[dr, "\tau"] & \\
 & & L
\end{tikzcd}$$

The free product $H_1 * H_2$ is a coproduct in Groups. Thus, there exists a homomorphism $\mu : H_1 * H_2 \longrightarrow L$ such that

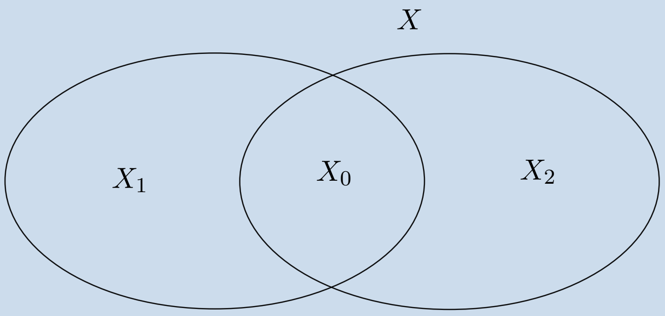
$$\begin{tikzcd}
 & H_1 * H_2 \arrow[dd, "\mu"] & \\
H_1 \arrow[ur] \arrow[dr, "\theta_1"'] & & H_2 \arrow[ul] \arrow[dl, "\theta_2"] \\
 & L &
\end{tikzcd}$$

Since $\theta_1 \circ \phi_1 = \theta_2 \circ \phi_2$, we have $N \leq \ker \mu$. Thus μ factors through $H_1 * H_2/N = K$. This is the desired morphism τ. □

4.1 The Seifert–van Kampen Theorem

We now have the necessary group theoretic tools required to state and prove the Seifert–van Kampen theorem, which allows us to "decompose" a topological space into smaller pieces, and compute the fundamental group of the full space in terms of the fundamental groups of the smaller pieces.

Theorem 4.10 (*Seifert–van Kampen*). Let $X = X_1 \cup X_2$, where X_1, X_2 are open. Assume X_1, X_2 and $X_0 = X_1 \cap X_2$ are nonempty and path connected.



Fix $p \in X_0$ and let the following be a commutative diagram

$$\begin{tikzcd}
& X_0 \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\
X_1 \arrow[dr, hook, "j_1"'] & & X_2 \arrow[dl, hook', "j_2"] \\
& X &
\end{tikzcd}$$

The fundamental group $\pi_1(X,p)$ is the pushout of the diagram

$$\begin{tikzcd}
\pi_1(X_0,p) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(X_1,p) \\
\pi_1(X_1,p) &
\end{tikzcd}$$

Proof. Firstly, $\pi_1(X,p)$ is a solution,

$$\begin{tikzcd}
\pi_1(X_0,p) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(X_1,p) \arrow[d, "\pi_1(j_1)"] \\
\pi_1(X_1,p) \arrow[r, "\pi_1(j_2)"] & \pi_1(X,p)
\end{tikzcd}$$

so this commutes since π_1 is a functor. Suppose G is a group and

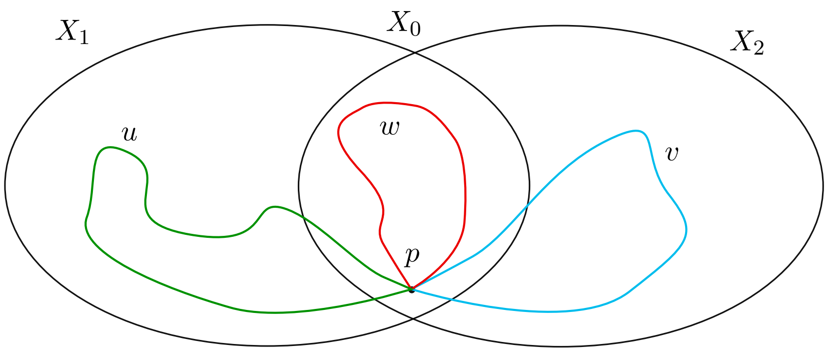
$$\begin{tikzcd}
\pi_1(X_0,p) \arrow[r] \arrow[d] & \pi_1(X_1,p) \arrow[d, "\phi_1"] \\
\pi_1(X_1,p) \arrow[r, "\phi_2"] & G
\end{tikzcd}$$

We need to construct a unique group homomorphism $\psi : \pi_1(X,p) \longrightarrow G$ such that

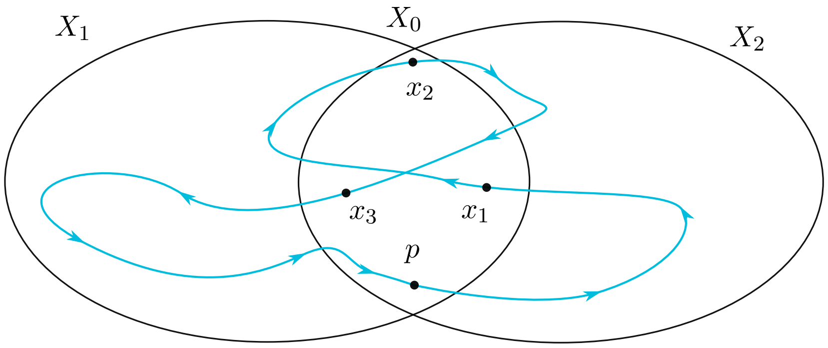
$$\begin{tikzcd}
\pi_1(X_0,p) \arrow[r] \arrow[d] & \pi_1(X_1,p) \arrow[d] \arrow[ddr, bend left, "\phi_1"] & \\
\pi_1(X_2,p) \arrow[r] \arrow[drr, bend right, "\phi_2"'] & \pi(X,p) \arrow[dr, "\psi"] & \\
& & G
\end{tikzcd}$$

Before proving the general version of the theorem, let us take a look at how ψ should behave with respect to loops when they are completely contained in either X_0, X_1 or X_2. We will then formulate an argument to reduce the general case to one of these simpler cases.

Let u be a loop whose image is entirely contained in X_1. We want to define $\psi([u]) \in G$. In this case we have no choice since we want the last diagram to commute. We must set $\psi([u]) := \phi_1([u])$. Similarly, if v is another loop entirely contained in X_2, we must define $\psi([v]) := \phi_2([v])$. What about a loop completely contained in X_0? Then we have two definitions for $\psi([w])$, does it happen that $\phi_1([w]) = \phi_2([w])$? Assuming the diagram commutes, we must have $\phi_1(\pi_1(i_1)[w]) = \phi_2(\pi_1(i_2)[w])$.



Now, for the general case, we start with an arbitrary loop u in X. We choose finitely many points $0 = s_0 < s_1 < ... < s_n = 1$ such that if $x_i = u(s_i)$ then each x_i lies in X_0 and $u_{|_{[s_1,s_{i+1}]}}$ is either contained in X_1 or X_2.



Note that such s_i exist due to the Lebesgue number lemma. Now let v_i be a reparametrisation on $[0,1]$ of $u_{|_{[s_i,s_{i+1}]}}$. Pick an arbitrary path w_i contained in X_0 with extremes $w_i(0) = p$ and $w_i(1) = x_i$. Then $\tilde{w}_i = w_{i-1} * v_i * \overline{w_i}$ is a loop entirely contained in X_1 or X_2. We can now factor u as the following concatenation of loops entirely contained in either X_1 or X_2.

$$u = \tilde{w}_1 * \tilde{w}_2 * ... * \tilde{w}_n.$$

In general, let ϕ_* to be either ϕ_1 or ϕ_2 depending as to whether $\tilde{w}_i$ lies in X_1 or X_2, and set

$$\psi([u]) = \prod \phi_*([\tilde{w}_i])$$

We now have to check whether out definition of ψ satisfies the following statements.

(i) Does ψ depend on the x_i's?

(ii) Does ψ depend on the w_i's?

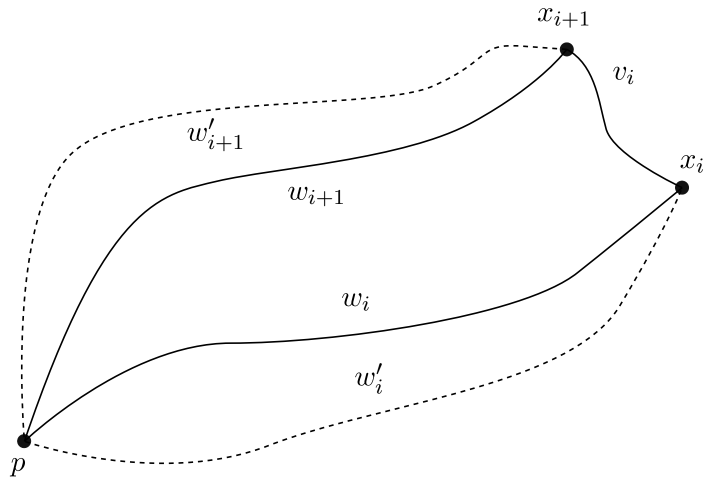
(iii) If $u \simeq v$ rel p then is $\psi(u) = \psi(v)$?

(iv) Is ψ a homomorphism?

(v) Does the diagram commute

(vi) Is ψ the unique map with these properties?

Suppose we have already dealt with (i)–(iii). Then the remaining properties are clear. ψ is a homomorphism by construction and the diagram commutes (from what we said at the start) and similarly, the remarks at the start of the proof show we have no choice in the definition of ψ. All we have to do is check ψ satisfies (i)–(iii).



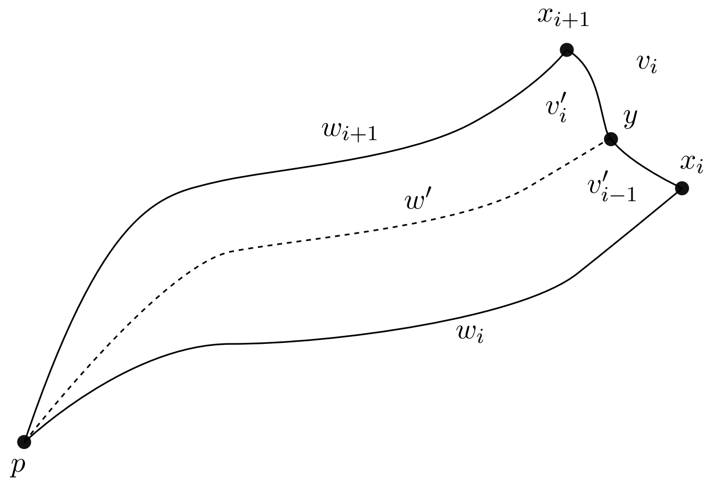
We want to show

$$\prod \phi_*([w_i * v_i * \overline{w_{i+1}}]) = \prod \phi_*([w'_i * v_i * \overline{w'_{i+1}}]).$$

This can be factored as follows,

$$\begin{aligned}&\phi_1\left([w_i * \overline{w'_i} * w'_i * v_i * \overline{w'_{i+1}} * w'_{i+1} * \overline{w_{i+1}}]\right) = \\ &= \phi_1\left([w_i * \overline{w'_i}i]\right) * \phi_1\left([w'_i * v_i * \overline{w'_{i+1}}]\right) * \phi_*\left([w'_{i+1} * \overline{w_{i+1}}]\right).\end{aligned}$$

And taking the product cancels everything out.



We now have to check that by adding an additional point y, the value of ψ will not be changed. Indeed, let w' be the path connecting p and y in X_0. Suppose that the loop $w_i * v_i * \overline{w_{i+1}}$ is contained in X_1. Then, if we denote by v'_{i-1} the part of the path v that connects x_i and y, and similarly, v'_i with y and x_{i+1}, the same is true for the two loops

$$w_i * v'_{i-1} * \overline{w'}, \quad w' * v'_i * \overline{w_{i+1}}.$$

Thus, we have

$$\phi_1\left([w_i * v'_{i-1} * \overline{w'}]\right)\phi_1\left([w' * v'_i * \overline{w_{i+1}}]\right) =$$

$$= \phi_1 \left([w_i * v'_{i-1} * \overline{w'} * w' * v'_i * \overline{w_{i+1}}] \right) =$$
$$= \phi_1 \left([w_i * v_i * \overline{w_{i+1}}] \right).$$

This shows that adding an additional point to the x_i's does not change the value of $\psi([u])$. More generally, the same is true if we add a finite number of new points. Now suppose we are given two different sets of points $\{x_i\}$ and $\{y_i\}$. Then by taking the union of both we can refine the set of points without changing $\psi([u])$. This shows that it does not depend on the choice of x_i.

It remains to show that ψ is defined on $\pi_1(X, p)$. Suppose $U : u \simeq v$ rel p. We divide $I \times I$ into a grid of squares such that each square is mapped by U into either X_1 or X_2. Such a decomposition exists by the Lebesgue number lemma. Proceeding one small rectangle at a time, this deforms u into v through a finite sequence of paths such that each step involves a homotopy in which the only change occurs in either X_1 or X_2. For such a restricted deformation, we may choose $\{x_i\}$ so that the value of ψ remains unchanged. This completes the proof.

□

There is a more general version of the Seifert–van Kompen theorem on grupoids.

Theorem 4.11. Let Grpd be the category of grupoids and $\Pi : \text{Top} \longrightarrow \text{Grpd}$. Let $\mathcal{U} = \{X_0, X_1, X_2\}$. Suppose $\mathcal{U}$ is an open cover of X with the property that the intersection of finitely many elements of $\mathcal{U}$ again belongs to $\mathcal{U}$. Then regard $\mathcal{U}$ as a category where the objects are the open sets $U \in \mathcal{U}$ and

$$\text{Hom}(U, V) = \{i : U \longrightarrow V\} \text{ if } U \subset V,\ \emptyset \text{ otherwise.}$$

Then the fundamental groupoid $\Pi(X)$ is the colimit of the diagram $\Pi_{|\mathcal{U}}$. In other words, $\Pi(X)$ is uniquely characterized by the following universal property.
Suppose $\mathcal{C}$ is a grupoid and suppose $\tau : \Pi \longrightarrow \mathcal{C}$ is a functor.

5 Singular homology

We can understand homology as the study of holes in topological spaces. The difficult problem about studying holes is that there is no straightforward definition of what a hole is. What we will do is think of topological spaces as being constructed out of simplices, and study the holes in these, since that is much easier.

But before we do this, we need to develop the algebraic tools that are needed for formalizing and handling the objects we just described.

5.1 Chain complexes

Definition 5.1. A chain complex consists of abelian groups C_n, $n \in \mathbb{Z}$, together with group homomorphisms $\partial_n : C_n \longrightarrow C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ for every $n \in \mathbb{Z}$.

$$\dots\ C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}\ \dots$$

By convention, we will normally write ∂ instead of ∂_n, and a chain complex will be abbreviated by $(C_\bullet, \partial)$.

For example, let (C_n) be any abelian group. Define a chain complex by setting $\partial = 0$. But why is $\partial^2 = 0$ a natural condition? When we have a mathematical object with symmetry, we can get to conditions of the type $\partial^2 = 0$. For example, for $f \in C^2$,

$$\frac{\partial^2}{\partial x^i \partial x^j} f = \frac{\partial^2}{\partial x^j \partial x^i} f,$$

and by substracting one from the other we get to a $\partial^2 = 0$ condition.

Now we make chain complexes into a category. Suppose $(C_\bullet, \partial)$ and $(D_\bullet, \delta)$ are two chain complexes. A chain map from $C_\bullet$ to $D_\bullet$ is a collection of group homomorphisms $f_n : C_n \longrightarrow D_n$ such that the following diagram commutes

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

for all $n \in \mathbb{Z}$. As with δ, we will usually write $f : C_\bullet \longrightarrow D_\bullet$ and omit the subscript n, so

$$\delta f = f \partial.$$

The category Comp has chain complexes as objects, chain maps as morphisms, and the composition is defined as

$$(f \circ g)_n = f_n \circ g_n.$$

Definition 5.2. Let $(C_\bullet, \partial)$ be a chain complex. We set

$$Z_n = Z_n(C_\bullet, \partial) = \ker \partial_n,$$

and note it is a subgroup of C_n. Elements of Z_n are called "cycles". We also set

$$B_n = B_n(C_\bullet, \partial) = \operatorname{Im} \partial_{n+1},$$

and note it is a subgroup of C_n too. Elements of B_n are called "boundaries".
The fact that $\partial_n \circ \partial_{n+1} = 0$ tells us that B_n is a subgroup of Z_n.This means that we can look at the quotient group

$$Z_n/B_n = H_n = H_n(C_\bullet, \partial).$$

H_n is the n–th homology group of the chain complex $C_\bullet$.

Oftentimes chain complexes are too "big". The idea of looking at the homology is a way of simplifying the picture. Now we introduce a bit of notation.

- The maps ∂ are usually called "boundary maps".
- If $c \in Z_n$ is a cycle, we write $\langle c \rangle$ to denote the element of H_n corresponding to c.

Suppose $f : (C_\bullet, \partial) \longrightarrow (D_\bullet, \delta)$ is a chain map. We claim that there is a well defined group homomorphism

$$H_n(f) : H_n(C) \longrightarrow H_n(D)$$

that sends $\langle c \rangle_C \mapsto \langle fc \rangle_D$. Since

$$\begin{tikzcd} C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[d, "f_n"] \\ D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \end{tikzcd}$$

we see that $f_n(Z_n(C)) \subset Z_n(D)$, that is, "$f$ maps cycles to cycles". Also, $f_n(B_n(C)) \subset B_n(D)$, and "$f$ maps boundaries to boundaries". Thus there is a well defined quotient map $H_n(f)$.

Proposition 5.3.

$$\begin{array}{rcl} H_n : \text{Comp} & \longrightarrow & \text{Ab} \\ C_\bullet & \longmapsto & H_n(C) \\ f & \longmapsto & H_n(f) \end{array}$$

is a functor. Also,

$$\begin{array}{rccc} \mathbb{H} : & \text{Comp} & \longrightarrow & \text{Comp} \\ & (C_\bullet, \partial) & \longmapsto & (H_n(C_\bullet), 0) \\ & f : C_\bullet \to D_\bullet & \longmapsto & \mathbb{H}(f) : (H_n(C_\bullet), 0) \to (H_n(D_\bullet), 0) \end{array}$$

is a functor and it is called the algebraic homology functor.

The goal now is to apply chain complexes to topological spaces in the following way.

Let X be a topological space. We will search for a way to associate a chain complex $C_\bullet(X)$ which encodes all the interesting topology of X. Once we have done this, we apply $\mathbb{H}$ to get its homology,

$$\text{Top} \longrightarrow \text{Comp} \xrightarrow{\mathbb{H}} \text{Comp}\,.$$

We will see that there are several options to choose from in order to go from Top to Comp: singular homology, cellular homology, simplicial homology, Morse homology, etc. The final theorem of the course will show that essentially, it does not matter which functor Top $\longrightarrow$ Comp we choose, the end result is the same.

5.2 Singular homology

Definition 5.4. Let F be an abelian group. A subset $B \subset F$ is said to be a basis if each subgroup $\langle b \rangle$ has infinite order for $b \in B$ and $F = \bigoplus_{b \in B} \langle b \rangle$.

Saying F has basis B means that for every $x \in F$, we can uniquely write

$$x = \sum_{b \in B} m_b \cdot b, \quad \text{where } m_b \in \mathbb{Z}$$

such that at most finitely many m_b's are non-zero.

Definition 5.5. The rank of F is the cardinality of a basis.

Note that there are two caveats to this definition.

- A basis has to exist.
- Any two bases have to have the same cardinality.

If B is a set, there exists a free abelian group with basis b.

$$F = \bigoplus_{b \in B} \mathbb{Z}_b.$$

Remark. If B is a group, one must make sure not to confuse relations in B with relations in F. For example, if $B = \mathbb{Z}$, then $2 + 3 = 5$. Let $F = \{(a_n) : n \in \mathbb{Z}\}$ with infinitely many non zero terms. We have $(a_n) + (b_n) = (a_n + b_n)$. However,

$$2 + 3 = 5 \not\Longrightarrow a_2 + a_3 = a_5.$$

If G is any abelian group, one can always find a free abelian subgroup F such that G/F is torsion and defining the rank of G as the rank of F is well defined.

Lemma 5.6. Let F be a free abelian group with basis B, and let $\phi : B \longrightarrow A$ be a function, where A is an abelian group. There exists a unique homomorphism $\tilde{\phi} : F \longrightarrow A$ extending ϕ

$$\begin{tikzcd}
F \arrow[dr, dotted, "\tilde{\phi}"] & \\
B \arrow[u, hook] \arrow[r, "\phi"] & A
\end{tikzcd}$$

by linearity,

$$\sum m_b \cdot b \longmapsto \sum m_b \cdot \phi(b).$$

Definition 5.7. Let $z_0, ... z_n$ be points in $\mathbb{R}^n$. We say $z_0, ..., z_n$ are affinely independent if $z_1 - z_0, ..., z_1 - z_n$ are linearly independent.
If $z_0, ..., z_n$ are affinely independent, we define the simplex with vertices $z_0, ..., z_n$ to be

$$[z_0 : ... : z_n] := \left\{ x \in \mathbb{R}^n \,:\, x = \sum_{i=0}^{n} s_i z_i,\ \sum_{i=0}^{n} s_1 = 1,\ s_i \geq 0 \right\}.$$

If $x \in [z_0 : ... : z_n]$ we call the tuple (s_i) the coordinates of x.

A simplex on $\mathbb{R}$ is a segment, and on $\mathbb{R}^2$ it is a triangle. If $z_0, ... z_n$ and $w_0, ..., w_n$ are affinely independent, then the simplices $[z_0 : ... : z_n]$ and $[w_0 : ... : w_n]$ are homeomorphic. Therefore, it is convenient to use the standard basis in $\mathbb{R}^n$.

Let e_i be the vector in $\mathbb{R}^n$ with a 1 in the $(i+1)$–st position,

$$e_0 = (1, 0, ..., 0)$$
$$...$$
$$e_{n-1} = (0, 0, ..., 1).$$

We let $\Delta^n = [e_0 : ... : e_n]$ be the standard n–simplex.

If $[z_0 : ... : z_n]$ is an n–simplex, then by deleting one of the z_i's, we obtain an $(n-1)$–simplex, $[z_0 : ... : \widehat{z_i} : ... : z_n]$, where $\widehat{z_i}$ means we delete z_i. Together these are the faces of the simplex.

Definition 5.8. A singular n–simplex in X is a continuous map $\sigma : \Delta^n \longrightarrow X$ (do not confuse this with $\sigma(\Delta^n)$, which is a subspace of X).

For each $p \in X$, there is a unique simplex σ_p in X given by $\sigma_p(y) = p$.

Definition 5.9. Let $C_n(X)$ be the free abelian group with basis all the singular simplices $\sigma : \Delta^n \longrightarrow X$.

For $n < 0$, we set $C_n = \emptyset$. An element $c \in C_n(X)$ can be uniquely written as $c = \sum m_i \sigma_i$ where $m_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \longrightarrow X$. One cannot regard c as being a map.

For example, suppose $X = \mathbb{R}^n$ and $\sigma : \Delta^n \longrightarrow \mathbb{R}^n$ is any simplex. Define τ as $\tau(x) = -\sigma(x) \in \mathbb{R}^n$. In other words, $\sigma(x) + \tau(x) \equiv 0$ for all $x \in \Delta^n$. Nevertheless, in $C_n(\mathbb{R}^n)$, σ and τ are basis elements and are thus linearly independent. This is

$$\sigma + \tau \in C_n(\mathbb{R}^n)$$

$$x \mapsto (\sigma + \tau)(x)$$

and $\sigma + \tau \neq \sigma + \tau$. We can consider

$$\begin{aligned} X &\longrightarrow C_n(X) \\ p &\longmapsto \sigma_p \end{aligned}$$

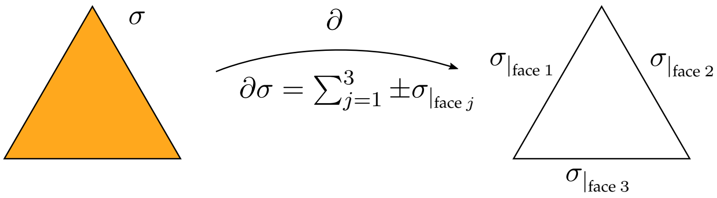
and it turns out that $C_n(X)$ is really large. We want to make $C_\bullet$ into a chain complex. That is, we want to define $\partial : C_n(X) \longrightarrow C_{n-1}(X)$.

Definition 5.10. We define the boundary maps

$$\partial : C_n(X) \longrightarrow C_{n-1}(X)$$

by setting

$$\partial\sigma = \sum_{j=0}^{n} (-1)^j \sigma_{|[e_0:\ldots:\widehat{e_j}:\ldots:e_n]}, \quad \sigma \in C_n.$$



Define homeomorphisms

$$\varepsilon_j^n : \Delta^{n-1} \longrightarrow j\text{–th face of the } \Delta^n.$$

Then $\partial\sigma = \sum_{j=0}^{n} (-1)^j \sigma \circ \varepsilon_j^n$.

There is a unique map $\partial : C_n \longrightarrow C_{n-1}$ by "extending by linearity".

Theorem 5.11. $(C_\bullet(X), \partial)$ is a chain complex.

Proof. We need only show that

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

is zero for all n. It suffices to show that for $\sigma : \Delta^{n+1} \longrightarrow X$ one has $\partial_n \partial_{n+1} \sigma = 0$. We will use the following face relation. Given

$$\varepsilon_j^n(s_o, \ldots, s_n) = (s_0, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_n),$$

then

$$\varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n = \varepsilon_j^{n+1} \circ \varepsilon_k^n : \Delta^{n-1} \longrightarrow j\text{–th face of } \Delta^n \longrightarrow (k+j)\text{–th face of } \Delta^{n+1}$$

whenever $k < j$. Now, we get

$$\partial_n \partial_{n+1} \sigma = \sum_{j=0}^{n+1} (-1)^j \partial_n \left(\sigma \circ \varepsilon_j^{n+1}\right) = \sum_{j \leq k} (-1)^{k+j} \sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n + \sum_{k<j} (-1)^{j+k} \sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n =$$

$$= \sum_{j\leq k}(-1)^{k+j}\sigma\circ\varepsilon_j^{n+1}\circ\varepsilon_k^n + \sum_{k<j}(-1)^{j+k}\sigma\circ\varepsilon_k^{n+1}\circ\varepsilon_{j-1}^n.$$

Using the face relation with $l = j-1$, the second term turns into

$$\sum_{k<l+1}(-1)^{k+j}\sigma\circ\varepsilon_j^{n+1}\circ\varepsilon_k^n,$$

which cancels with the first term and thus $\partial_n\partial_{n+1}\sigma = 0$. ☐

This defines $C_\bullet$ on objects of Top. Suppose $f : X \longrightarrow Y$ is a continuous function. We want to define

$$C(f) : C(X) \longrightarrow C(Y),$$

which we will call $f_\#$. It has to be a chain map such that

$$\sigma : \Delta^n \longrightarrow X \xrightarrow{f} Y$$

$$f_\#(\sigma) = f\circ\sigma,$$

and we define it by extending by linearity, that is,

$$f_\#\left(\sum m_\sigma\sigma\right) = \sum m_\sigma f(\sigma)$$

It is immediate that

(i) $id_\# = id_{C_\bullet(X)}$.

(ii) $(f\circ g)_\# = f_\#\circ g_\#$.

Putting this together we find there is a well defined series of functors H_n, since if f is continuous, then it is easy to check that $f_\#(Z_n(X)) \subset Z_n(Y)$ and $f_\#(B_n(X)) \subset B_n(Y)$. This is,

$$\begin{array}{rccc} H_n : & \text{Top} & \longrightarrow & \text{Ab} \\ & X & \longmapsto & H_n(X) \\ & f : X \to Y & \longmapsto & \begin{array}{c} H_n(f) : H_n(X) \longrightarrow H_n(Y) \\ \langle c\rangle \longmapsto \langle f_\# c\rangle . \end{array} \end{array}$$

The goal now is to show that homology functors H_n factor to define functors hTop $\longrightarrow$ Ab. This will give us the proposition that we used in order to prove Brouwer's fixed point theorem.

Proposition 5.12.

$$H_n(B^i) = 0 \quad \forall n > 0 \text{ and } H_0(B^i) = \mathbb{Z}.$$

Proof. By the assumption that H_n factors to hTop, we have

$$H_n(B^i) = H_n(\{*\}),$$

where $\{*\}$ is the topological space with one point. The computation of $H_n(\{*\})$ is an easy exercise. ☐

Let us return to Comp and the algebraic homology functor $\mathbb{H} : \text{Comp} \longrightarrow \text{Comp}$. Suppose $C_\bullet$ and $D_\bullet$ are two chain complexes. Let

$$f, g : C_\bullet \longrightarrow D_\bullet$$

be two chain maps,

$$H_\bullet(f), H_\bullet(g) : H_\bullet(C) \longrightarrow H_\bullet(D).$$

When is $H_\bullet(f) = H_\bullet(g)$?

Definition 5.13 (*Chain Homotopy*). A chain homotopy P from a chain map f to a chain map g is a collection of group homomorphisms

$$P_n : C_n \longrightarrow D_{n+1}$$

such that

$$\delta_{n+1} \circ P_n - P_{n-1} \circ \partial_n = f_n - g_n \quad \forall n \in \mathbb{Z}.$$

Lemma 5.14. If such a P exists, then $H_\bullet(f) = H_\bullet(g)$.

Proof. Take $c \in Z_n(C)$. Then

$$f_n c - g_n c = \delta P c - P \partial c.$$

Since c is a cycle, $\partial c = 0$, and $\delta P c \in B_n(D)$. Thus,

$$H_n(f) \langle c \rangle = \langle f_n c \rangle = \langle g_n c \rangle = H_n(g) \langle c \rangle .$$

□

We write $P_\bullet : f_\bullet \simeq g_\bullet$ to indicate the existence of such a chain homotopy. At the moment, we have to complete the construction

$$\text{Top} \xrightarrow{C_\bullet} \text{Comp} \xrightarrow{\mathbb{H}} \text{Comp} .$$

Theorem 5.15. Let $f, g : X \longrightarrow Y$ be continuous functions between topological spaces. If $f \simeq g$ then $H_\bullet(f) = H_\bullet(g)$.

Proof. We have to construct $P : f_\# \simeq g_\#$. However, we do this only in a very special case. Let

$$\begin{aligned} i_0 : X &\longrightarrow X \times [0,1] \\ x &\longmapsto (x, 0) \end{aligned}$$

and

$$\begin{aligned} i_1 : X &\longrightarrow X \times [0,1] \\ x &\longmapsto (x, 1) \end{aligned} .$$

We construct $P : i_{0\#} \simeq i_{1\#}$. This is sufficient, since we already know $H_\bullet$ is a functor. Indeed, suppose $F : f \simeq g$. Then

$$f = F \circ i_0, \quad g = F \circ i_1$$

and

$$H_\bullet(f) = H_\bullet(F \circ i_0) = H_\bullet(F) \circ H_\bullet(i_0) = H_\bullet(F) \circ H_\bullet(i_1) = H_\bullet(F \circ i_1) = H_\bullet(g).$$

Thus it remains to construct such a P. We will not do this, since it would be necessary to prove two more results that we will use just once. Also, by the end of the course we will give a proof for this result which is only a couple of lines in length. □

This "completes" the construction of singular homology. The advantage of homology is that it is much easier to compute thatn π_j. In fact, computing $\pi_i(\mathbb{S}^n)$ is an open problem for some j, n. On the other hand, computing $H_j(\mathbb{S}^n)$ is easy.

The reason why we use simplices is because it is easy to understand what the face of a simplex is. It is also possible to construct singular homology from cubes, which are easier to integrate on. However, it is very non–standard. Alternatively, there is the approach of trying to subdivide the topological space into simplices or cubes, making the resulting chain complex much smaller. However, this only works with certain topological spaces, it does not work in generally abstract topological spaces. It is what is known as simplicial homology. A newer version of it is cellular homology.

Remark. A 1–simplex is a path. In particular, a loop in X is a 1–simplex that starts and ends in the same place. In particular, if σ is a loop, then $\partial\sigma = 0$. If $u : \mathbb{S}^1 \longrightarrow X$, then $\langle u \rangle \in H_1(X)$. Let us consider

$$\begin{aligned} \pi_1(X,p) &\longrightarrow H_1(X) \\ [u] &\longmapsto \langle u \rangle \,. \end{aligned}$$

One could ask if this map is an isomorphism. It is clear that, since they are different categories, this is not the case in general. It suffices to choose $\pi_1(X,p)$ so that it is not abelian. Nevertheless, we will see that there is a functor

$$\begin{aligned} \mathrm{Ab} : \mathrm{Groups} &\longrightarrow \mathrm{Ab} \\ G &\longmapsto G^{\mathrm{Ab}} \end{aligned}$$

that "abelianizes" groups. With it, we will be able to prove that

$$\pi_1(X,p)^{\mathrm{Ab}} = H_1(X).$$

This shows that homology is "simpler" than homotopy. We will now study H_0 and H_1.

5.3 H_0 and H_1

First, let us introduce the convention that paths can be defined over 1–simplices. Indeed, since $I = [0,1]$ and Δ^1 are homeomorphic, we set $u : I \longrightarrow X$ and $u' : \Delta^1 \longrightarrow X$, and $u = u'$, but regarded as a 1–simplex. In fact, we would have $u'(1-s, s) = u(s)$. Throughout this section, we will identify $\sigma_p : \Delta^0 \longrightarrow X$ with p.

Proposition 5.16. If X is a path connected topological space, then $H_0(X) \cong \mathbb{Z}$ and moreover, $\langle p \rangle$ generates $H_0(X)$ for all $p \in X$.

Proof. $C_0(X)$ is the free abelian group generated by points of X. Remember that if $x, y \in X$, then $x + y$ is defined as the sum of $\sigma_x + \sigma_y$ in $C_0(X)$, but not in X. By definition, we have $C_{-1}(X) = 0$. Thus $Z_0(X) = \ker \partial : C_0 \longrightarrow C_{-1}$, so $Z_0(X) = C_0(X)$. Now we have to compute the boundaries. For that, note that an element of C_0 is of the form $c = \sum m_x x$ for $x \in X$ and $m_x \in \mathbb{Z}$, only finitely many non zero. We claim that

$$B_0(X) = \left\{c = \sum m_x x : \sum m_x = 0\right\}.$$

Suppose $c = \sum m_i x_i$ satisfies $\sum m_i = 0$. Fix $p \in X$ and paths $u_i : p \longrightarrow x_i$. Then u_i' is a 1–simplex with $u_i'(e_1) = x_i$ and $u_i'(e_0) = p$.

$$\partial u_i' = u_i'(e_1) - u_i'(e_0) = x_i - p.$$

Now consider $a = \sum m_i u_i' \in C_1(X)$.

$$\partial a = \partial\left(\sum m_1 u_i'\right) = \sum m_i(x_i - p) = \sum m_i x_i - \left(\sum m_i\right) p = c.$$

Thus $\partial a = c$, so $c \in B_0(X)$.

Suppose $d \in B_0(X)$. Then there exists $b \in C_1(X)$ such that $\partial b = d$. Suppose $b = \sum n_i \sigma_i$ for $n_i \in \mathbb{Z}$, $\sigma_i : \Delta^1 \longrightarrow X$. Thus

$$d = \partial b = \sum n_i\left(\sigma_i(e_1) - \sigma_i(e_0)\right) = \sum n_i \sigma_i(e_1) - \sum n_i \sigma_i(e_0),$$

and thus the sum of the coefficients in d is 0 since every coefficient in the expansion of d over the base appears twice with opposite signs. This proves the claim. Thus the map

$$\begin{aligned} \phi : Z_0(X) &\longrightarrow \mathbb{Z} \\ \sum m_x x &\longmapsto \sum m_x \end{aligned}$$

has kernel $B_0(X)$ and $H_0(X) \cong \mathbb{Z}$.

Suppose $x, y \in X$. Pick a path u from x to y. Then $\partial u' = y - x$, i.e., $\langle x\rangle = \langle y\rangle$ in homology. Suppose $c = \sum m_i x_i$ is an arbitrary generator of $H_0 = \mathbb{Z}\langle c\rangle$. Then $\phi\left(\sum m_i x_i\right)$ is a generator of $\mathbb{Z}$, i.e., $\phi(c) = \pm 1$. Replacing c by $-c$ if needed, assume $\phi(c) = 1$. Then $c - p \in C_0$ for any arbitrary p has $\phi(c - p) = 0$, so $\langle c\rangle = \langle p\rangle$. ☐

Lemma 5.17. If X has path components X_α then $H_\bullet(X) = \bigoplus_\alpha H_\bullet(X)$.

Corollary 5.18.

$$H_0 \cong \mathbb{Z}^{|\pi_0(X)|}.$$

This finishes the topics we wanted to cover about H_0. Now for H_1.

Definition 5.19. Fix $p \in X$ and define a map $h = h_p$ as

$$\begin{aligned} h_p : \pi_1(X, p) &\longrightarrow H_1(X) \\ [u] &\longmapsto \langle u'\rangle . \end{aligned}$$

This is well defined. Firstly, u' is a cycle since

$$\partial u' = u'(e_1) - u'(e_0) = u(1) - u(0) = p - p = 0.$$

Next, we show that if $[u]=[v]$ then $\langle u'\rangle = \langle v'\rangle$. If u is a loop in x then we alternatively regard u as being a map $\mathbb{S}^1 \longrightarrow X$. Call this map $\widehat{u}$. We have

$$u' = u\circ\theta : \Delta^1 \longrightarrow X,$$

where $\theta : \Delta^1 \longrightarrow I$ is a homeomorphism. Ehen u is a loop,

$$u = \widehat{u}\circ\tau, \qquad \begin{aligned} \tau : I &\longrightarrow \mathbb{S}^1 \\ s &\longmapsto e^{2\pi i s}. \end{aligned}$$

With this,

$$\begin{aligned} u' &= \widehat{u}\circ\tau\circ\theta, & \langle u'\rangle &= H(\widehat{u})\,\langle \tau\circ\theta\rangle \\ v' &= \widehat{v}\circ\tau\circ\theta, & \langle v'\rangle &= H(\widehat{v})\,\langle \tau\circ\theta\rangle\,. \end{aligned}$$

We already know that $H_\bullet$ is a functor on hTop, and since $[u]=[v]$, then $\widehat{u}$ and $\widehat{v}$ are homotopic rel I in $\mathbb{S}^1$. Therefore, $H(\widehat{u}) = H(\widehat{v})$

$$\langle u'\rangle = H(\widehat{u})\,\langle \tau\circ\theta\rangle = H(\widehat{v})\,\langle \tau\circ\theta\rangle = \langle v'\rangle\,.$$

We have now proved that

$$h : \pi_1 \longrightarrow H_1$$

is well defined, and we will now show that h is a group homomorphism. This means we want to see that

$$h\left([u]\right) + h\left([v]\right) = h\left([u*v]\right).$$

Define

$$(u*v)'(s_0,s_1) = \begin{cases} u'(2s_0-1, 2s_1) & s_1 \le 1/2 \\ v'(2s_0, 1-2s_1) & s_1 \ge 1/2. \end{cases}$$

We want to find an element c of C_2 such that $\partial c = u' + v' - (u*v)'$. Notice that by doing this, then the right hand side is in the image of ∂ and by taking equivalence classes, this goes to zero and we prove h is a homomorphism. However, we can find something even better, namely that there exists a continuous map $\sigma : \Delta^2 \longrightarrow X$ such that $\partial\sigma = u' + v' - (u*v)'$. Recall we denote by $\varepsilon_0, \varepsilon_1, \varepsilon_2$ the face maps. Thus if $\sigma : \Delta^2 \longrightarrow X$, then

$$\partial\sigma = \sigma\circ\varepsilon_0 - \sigma\circ\varepsilon_1 + \sigma\circ\varepsilon_2 = \sigma_0 - \sigma_1 + \sigma_2.$$

We want

$$\sigma(s_0,s_1,s_2) = (u*v)'\left(s_0 + \frac{s_1}{2}, s_2 + \frac{s_1}{2}\right),$$

and it is left to see that this choice yields the result. Conversely, we have the following lemma.

Lemma 5.20. If $\sigma : \Delta^2 \longrightarrow X$ is a 2–simplex in X,

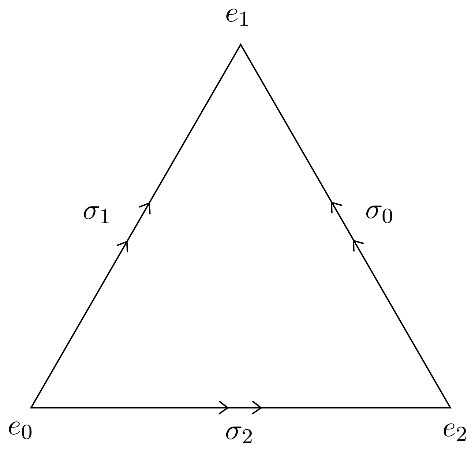
$$\sigma_0 = \sigma\circ\varepsilon_0, \quad \sigma_1 = \sigma\circ\varepsilon_1, \quad \sigma_2 = \sigma\circ\varepsilon_2,$$

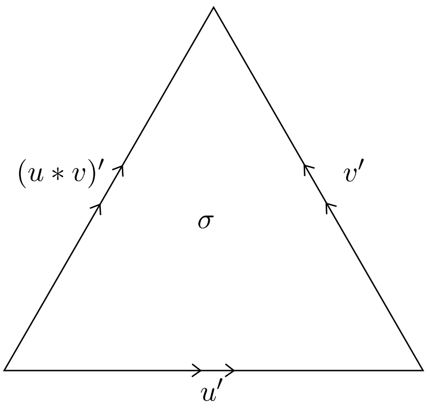
then the loop $\sigma_0 * \overline{\sigma_1} * \sigma_2$ is nullhomotopic, $[\sigma_0 * \overline{\sigma_1} * \sigma_2] = 0$.

Proof. To show this, recall that a continuous map $f : \mathbb{S}^1 \longrightarrow X$ is homotopic if and only if there exists $g : B^{n+1} \longrightarrow X$ such that $g|_{\partial B^{n+1}} = f$. Choose a homeomorphism

$$\varphi : (\Delta^2, \partial\Delta^2) \longrightarrow (B^2, \mathbb{S}^1)$$

and apply the previous remark. □





Lemma 5.21 (*Duck lemma*). Suppose F is a free abelian group with basis B and A is any abelian group. Suppose $b_0, ..., b_n$ are some elements of B. Suppose $a_0, ...a_n$ are some elements of A such that if $b_i = b_j$, then $a_i = a_j$. If $\sum m_i b_i = 0$ then $\sum m_i a_i = 0$.

Proof. Define

$$\begin{array}{rcl} \varphi : & B & \longrightarrow A \\ & b_i & \longmapsto a_i \\ & b \in B \setminus \{b_0, ..., b_n\} & \longmapsto 0. \end{array}$$

Then φ extends uniquely to $\tilde{\varphi} : F \longrightarrow A$ by linearity, and this is a homomorphism,

$$\tilde{\varphi}\left(\sum m_i b_i\right) = \sum m_i a_i.$$

□

Basically, the Duck lemma tells us that if the b_i's satisfy a relation in F, then the a_i's satisfy "the same relation" in A.

> *"If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck."*

We now introduce abelianisation. Let us consider the following universal property.

Definition 5.22. Let G be a group and A be an abelian group. $\psi : G \longrightarrow A$ is a homomorphism. We look for an abelian group H with homomorphism $\varphi : G \longrightarrow H$.

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\varphi"'] & A \\ H \arrow[ur, "\exists! \text{ s.t. commutes}"'] & \end{tikzcd}$$

This property defines abelianisation,

$$(\cdot)^{\text{Ab}} : \text{Groups} \longrightarrow \text{Ab}\,.$$

Then it is not difficult to check that such a functor exists. Let $[G, G]$ be the commutator of G. Then $G/[G, G]$ satisfies the universal property. Call this the abelianisation of G, written G^{Ab}. Apply this to π_1,

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "h_p"] \arrow[d] & H_1(X) \\
\pi_1(X,p)^{\mathrm{Ab}} \arrow[ur, "\eta_p"'] &
\end{tikzcd}$$

Theorem 5.23 (*Hurewicz*). Assume X is path connected. Then η_p in the above diagram is a group isomorphism and $H_1(X)$ is the abelianisation of $\pi_1(X,p)$.

Proof. We begin by proving η_p is surjective, which is automatically given by the fact that h_p is surjective. For each point $x \in X$, let w_x be a path $p \mapsto x$. Assume w_p is the constant path. Let $c \in Z_1$ be an arbitrary cycle. We find an element of π_1^{Ab} such that $\eta_p(\cdot) = \langle c \rangle$.

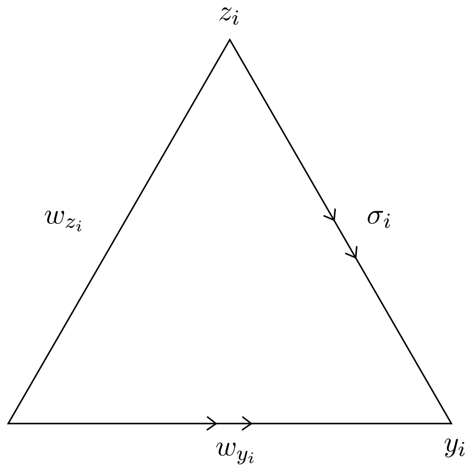
$$\begin{aligned}
c &= \sum m_i \sigma_i \\
0 = \partial c &= \sum m_i \left(\sigma_i(e_1) - \sigma_i(e_0)\right) = \sum m_i \left(y_i - z_i\right).
\end{aligned}$$

Apply the Duck lemma with

$$\begin{gathered}
F = C_0(X), \quad B = \{\text{points of } X\}, \quad A = C_1(X) \\
y_1, z_1, \dots, y_k, z_k \\
w'_{y_1}, w'_{z_1}, \dots, w'_{y_k}, w'_{z_k}.
\end{gathered}$$

Thus $\sum m_i \left(w'_{y_i} - w'_{z_i}\right) = 0$ in $C_1(X)$.

$$c = c - 0 = \sum m_i \sigma_i + \sum m_i (w'_{y_i} - w'_{z_i}) = \sum m_i \left(w'_{y_i} + \sigma_i - w'_{z_i}\right)$$



Thus $w_{z_i} * \sigma * \overline{w_{y_i}}$ is a loop. Since we already know h is a group homomorphism,

$$h_p\left(\prod [w_{z_i} * \sigma_i * \overline{w_{y_i}}]^{m_i}\right) = \sum m_i h\left([w_{z_i} * \sigma_i * \overline{w_{y_i}}]\right) = \sum m_i \left\langle w'_{z_i} + \sigma'_i - w'_{y_i} \right\rangle = \langle c \rangle .$$

This gives surjectivity of h_p and thus surjectivity of η_p. The next step is to construct an inverse for η_p to show that it is an isomorphism. We will not do this. $\square$

6 Homological algebra

Definition 6.1. Let A, B, C be abelian groups, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ homomorphisms. We say the sequence is exact (or exact at B) if $\text{Im}\, f = \ker g$.

For example, a homomorphism $f : A \longrightarrow B$ is injective if and only if $0 \longrightarrow A \xrightarrow{f} B$ is exact. Also, f is surjective if and only if $A \xrightarrow{f} B \longrightarrow 0$ is exact. In conjunction, f is an isomorphism if and only if $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ is exact.

If (A_n) is a collection of abelian groups with homomorphisms $f_n : A_n \longrightarrow A_{n-1}$, then we say the sequence

$$\dots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \dots$$

is exact if each chunk is exact, that is, $\ker f_n = \text{Im}\, f_{n+1}$.

Lemma 6.2. If $(C_\bullet, \partial)$ is a chain complex, then $H_\bullet(C, \partial) = 0$ if and only if the sequence is exact.

Proof. It is always true that $\ker \partial_n \supset \text{Im}\, \partial_{n+1}$, and $H_n = 0$ if and only if $\ker \partial_n = \text{Im}\, \partial_{n+1}$. ☐

This lemma tells us that if every cycle is the image of a previous simplex then we are not generating new cycles, which means that there can be no holes. Conversely, if there are not any holes then every cycle must be in the image of the boundary map.

Suppose $(C^n_\bullet, \partial)$ is a collection of chain complexes. Suppose $f^n_\bullet : C^n \longrightarrow C^{n-1}$ is a collection of chain maps.

$$\begin{tikzcd}
C^{n+1}_{i+1} \arrow[r, "\partial^{n+1}_{i+1}"] \arrow[d, "f^{n+1}_{i+1}"'] & C^{n+1}_i \arrow[r, "\partial^{n+1}_i"] \arrow[d, "f^{n+1}_i"] & C^{n+1}_{i-1} \arrow[d, "f^{n+1}_{i-1}"] \\
C^n_{i+1} \arrow[r, "\partial^n_{i+1}"] \arrow[d, "f^n_{i+1}"'] & C^n_i \arrow[r, "\partial^n_i"] \arrow[d, "f^n_i"] & C^n_{i-1} \arrow[d, "f^n_{i-1}"] \\
C^{n-1}_{i+1} \arrow[r, "\partial^{n-1}_{i+1}"] & C^{n-1}_i \arrow[r, "\partial^{n-1}_i"] & C^{n-1}_{i-1}
\end{tikzcd}$$

We say $(f^n_\bullet, C^n_\bullet, \partial^n)$ is an exact sequence of chain complexes if and only if all columns are exact.

Definition 6.3. A short exact sequence is one of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

A long exact sequence is a (possibly) infinite one.

For chain complexes, a short exact sequence of chain complexes is of the form

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0.$$

Theorem 6.4 (*Long exact sequence*). Suppose $0 \longrightarrow C_\bullet \xrightarrow{f_\bullet} C'_\bullet \xrightarrow{g_\bullet} C''_\bullet \longrightarrow 0$ is a short exact sequence of chain complexes. Then there are homomorphisms

$$\delta_n : H_n(C''_\bullet) \longrightarrow H_{n-1}(C'_\bullet)$$

such that there is a long exact sequence

$$\ldots \longrightarrow H_n(C_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(C'_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(C''_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \xrightarrow{H_{n-1}(f_\bullet)} H_{n-1}(C'_\bullet)\ldots$$

Lemma 6.5 (*The five lemma*). Suppose we have a commutative diagram of abelian groups such that the rows are exact

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "i"] & E \arrow[d, "j"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

If f, g, i, j are isomorphisms, then so is h.

To prove this result, we use the same technique as the one required to prove the next result, called "diagram chasing".

Lemma 6.6 (*The snake lemma*). Assume the following diagram commutes and its rows are exact.

$$\begin{tikzcd}
 & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' &
\end{tikzcd}$$

Then there is a well defined homomorphism $\delta : \ker h \longrightarrow \operatorname{coker} f$ such that the following is exact.

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h.$$

Proof. Note that this proof is quite lengthy since there are many things to do. However, as you will see, it is not difficult.

In the diagram below, $k = i|_{\ker f}$ is the induced map, and so are l, p, q, and p satisfies

$$\begin{aligned} p : \quad \operatorname{coker} f &\longrightarrow \quad \operatorname{coker} g \\ a' + \operatorname{Im} f &\longmapsto i'(a') + \operatorname{Im} g. \end{aligned}$$

We want to prove that both the top and bottom rows are exact. First, we note that $\operatorname{Im} k \subset \ker g$, since

$$g(k(a)) = i'(f(a)) = 0$$

as $a \in \ker f$.

Now we check that $\operatorname{Im} k = \ker l$. Indeed,

$$l \circ k(a) = j \circ i(a) = 0$$

by exactness at B, so $\operatorname{Im} k \subset \ker l$. Conversely, if $l(b) = 0$ then $j(b) = 0$ and $b = i(a)$ for some $a \in A$ by exactness at B. Moreover $i'(f(a)) = g(i(a)) = g(b) = 0$ as $b \in \ker g$. Since i' is injective, it follows $f(a) = 0$ and thus $a \in \ker f$ with so that $k(a) = b$. Thus $\ker l \subset \operatorname{Im} k$ and we have exactness at $\ker g$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker f \arrow[r, "k"] \arrow[d] & \ker g \arrow[r, "l"] \arrow[d] & \ker h \arrow[d] \arrow[ddll, out=-30, in=150] & \\
 & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"] \arrow[d] & B' \arrow[r, "j'"] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker} f \arrow[r, "p"] \arrow[d] & \operatorname{coker} g \arrow[r, "q"] \arrow[d] & \operatorname{coker} h \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let us now prove exactness at $\operatorname{coker} g$. The composition $q \circ p$ is obviously zero since $j' \circ i' = 0$ by exactness at B',

$$q \circ p(a') = j' \circ i'(a') + \operatorname{Im} h = 0 + \operatorname{Im} h = 0 \in \operatorname{coker} h.$$

Thus $\operatorname{Im} p \subset \ker q$. Conversely, suppose $q(b' + \operatorname{Im} g) = 0$. This means that $j'(b') \in \operatorname{Im} h$, so there exists $c \in C$ such that $h(c) = j'(b')$. Since j is surjective, there exists $b \in B$ such that $j(b) = c$. Now observe $j'(b' - g(b)) = j'(b') - j'(g(b)) = h(c) - h(j(b)) = h(c) - h(c) = 0$. Thus $b' - g(b) \in \operatorname{Im} i'$ by exactness at B'. If a' is such that $i'(a') = b' - g(b)$, then

$$p(a' + \operatorname{Im} f) = i'(a') + \operatorname{Im} g = b' - g(b) + \operatorname{Im} g = b' + \operatorname{Im} g.$$

Thus $\ker q \subset \operatorname{Im} p$, which proves exactness at $\operatorname{coker} g$.

Now we give the definition of δ and prove that it is well defined. Suppose $c \in \ker h$. Since j is surjective, there exists $b \in B$ such that $j(b) = c$. In fact $g(b)$ lies in the kernel of j'.

$$j'(g(b)) = h(j(b)) = h(c) = 0, \quad \text{since } c \in \ker h.$$

By exactness at B', there exists a unique $a' \in A'$ (since i' is injective) such that $i'(a') = g(b)$. Define

$$\delta(c) = (a' + \operatorname{Im} f) \in \operatorname{coker} f.$$

Since b is not unique, let us suppose that $b_1 \in B$ is another element such that $j(b_1) = c$. We obtain $a'_1 \in A'$. Let us check that

$$a' + \operatorname{Im} f = a'_1 + \operatorname{Im} f$$

as elements of $\operatorname{coker} f$. That is, there exists $a \in A$ such that $f(a) = a' - a'_1$. Since $j(b) = j(b_1) = c$, we have $b - b_1 \in \ker j$. Now we use exactness, so there exists $a \in A$ such that $i(a) = b - b_1$.

$$i'(f(a)) = g(i(a)) = g(b - b_1) = g(b) - g(b_1).$$

Since i' is injective, we have $f(a) = a' - a'_1$. This shows that δ is well defined and

$$\delta(c) = (i')^{-1} \circ g \circ j^{-1}(c) + \operatorname{Im} f.$$

Finally, let us check exactness at the two new places: $\ker h$ and $\operatorname{coker} f$. It is clear that $\operatorname{Im} l \subset \ker \delta$. Indeed, if $c = l(b)$ for some b then we can choose this b in the definition of δ. Then $g(b) = 0$ and hence the unique preimage under i' is $a' = 0$. Then $\delta(c) = 0 + \operatorname{Im} f = 0 \in \operatorname{coker} f$. Now suppose that $\delta(c) = 0$. This means that the element a' we found belongs to the image of f, say $a' = f(a)$. Then $g(i(a)) = i'(a') = g(b)$. Thus $b - i(a) \in \ker g$. Moreover $j(b - i(a)) = j(b) - j(i(a)) = c$ by exactness. This means that $l(b - i(a)) = c$ and thus $c \in \operatorname{Im} l$. This proves exactness at $\ker h$.

Now we check exactness at $\operatorname{coker} f$. Again, one direction is immediate, $p(\delta(c))$ is just the coset $i'(a') + \operatorname{Im} g$ in $\operatorname{coker} g$. But since $i'(a') = g(b)$, this coset is zero, and hence $\operatorname{Im} \delta \subset \ker p$. Conversely, suppose $p(a' + \operatorname{Im} f) = 0$. This means that $i'(a') \in \operatorname{Im} g$, so there exists $b \in B$ such that $g(b) = i'(a')$. Set $c = j(b)$. Then $h(c) = h(j(b)) = j'(g(b)) = j'(i'(a')) = 0$ by exactness at B'. Then by construction, $\delta(c) = a' + \operatorname{Im} f$. Thus $\ker p \subset \operatorname{Im} \delta$. This finally completes the proof. ☐

We now have the tools required to prove theorem 6.4.

proof of 6.4. Let $H_n = Z_n/B_n$, $H'_n = Z'_n/B'_n$ and $H''_n = Z''_n/B''_n$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H_n \arrow[r, "k"] \arrow[d] & H'_n \arrow[r, "l"] \arrow[d] & H''_n \arrow[d] & \\
 & C_n/B_n \arrow[r, "f_n"] \arrow[d, "\partial"] & C'_n/B'_n \arrow[r, "g_n"] \arrow[d, "\partial'"] & C''_n/B''_n \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & Z'_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & Z''_{n-1} \arrow[d] & \\
 & H_{n-1} \arrow[r] \arrow[d] & H'_{n-1} \arrow[r] & H''_{n-1} & \\
 & 0 & & &
\end{tikzcd}$$

The snake lemma provides $\delta_n : H''_n \longrightarrow H_{n-1}$ making the snake exact. ☐

6.1 Naturality of long exact sequences

Theorem 6.7. Suppose we have a commuting diagram of short exact sequences of chain complexes.

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "f_\bullet"] \arrow[d, "i_\bullet"] & B_\bullet \arrow[r, "g_\bullet"] \arrow[d, "j_\bullet"] & C_\bullet \arrow[r] \arrow[d, "k_\bullet"] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r, "f'_\bullet"] & B'_\bullet \arrow[r, "g'_\bullet"] & C'_\bullet \arrow[r] & 0
\end{tikzcd}$$

Then, the long exact sequences induced are such that the following diagram commutes.

$$\begin{tikzcd}
\ldots \arrow[r] & H_n(A) \arrow[r, "H_n(f)"] \arrow[d, "H_n(i)"] & H_n(B) \arrow[r, "H_n(g)"] \arrow[d, "H_n(j)"] & H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(k)"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(i)"] & \ldots \\
\ldots \arrow[r] & H_n(A') \arrow[r, "H_n(f')"] & H_n(B') \arrow[r, "H_n(g')"] & H_n(C') \arrow[r, "\delta'_n"] & H_{n-1}(A') \arrow[r] & \ldots
\end{tikzcd}$$

In words, saying "operation X on chain complexes is natural" is applying X to something that commutes still commutes.

Proof. Take $\langle c\rangle \in H_n(C)$. Since $g_n : B_n \longrightarrow C_n$ is surjective, choose b such that $g_n(b) = c$.

$$H_{n-1}(i)\circ\delta_n\langle c\rangle = H_{n-1}(i)\circ\delta_n\langle g_n(b)\rangle = H_{n-1}(i)\left\langle f_{n-1}^{-1}\partial b\right\rangle = \left\langle i_{n-1}\circ f_{n-1}^{-1}\partial b\right\rangle.$$

We also have

$$\begin{aligned}
k_{n-1}\circ g_{n-1} &= g'_{n-1}\circ j'_{n-1}\\
j_{n-1}\circ f_{n-1} &= f'_{n-1}\circ i_{n-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\langle i_{n-1}\circ f_{n-1}^{-1}\partial b\right\rangle = \left\langle (f'_{n-1})^{-1}\circ j_{n-1}\partial b\right\rangle = \left\langle (f'_{n-1})^{-1}\partial' j_{n-1} b\right\rangle = \\
= \delta'_n\left\langle g'_n\circ j_n(b)\right\rangle = \delta'_n\langle k_n\circ g_n(b)\rangle,
\end{aligned}$$

which means

$$H_{n-1}(i)\circ\delta_n\langle c\rangle = \delta'_{n-1}\circ H_n(k)\langle c\rangle.$$

□

Definition 6.8. Let $(C_\bullet, \partial)$ be a chain complex. A subcomplex $A_\bullet$ of $C_\bullet$ is a collection of subgroups $A_n \subset C_n$ such that $\partial(A_n) \subset A_{n-1}$. Thus $(A_\bullet, \partial|_A)$ is a chain complex in its own right.

If $A_\bullet$ is a subcomplex of $C_\bullet$, if we set $B_n := C_n/A_n$, then ∂ factors to define a boundary operator on $B_\bullet$. In this case,

$$0 \longrightarrow A_\bullet \longrightarrow C_\bullet \longrightarrow B_\bullet \longrightarrow 0$$

is exact, where the maps are the inclusion and the projection over the quotient.

Corollary 6.9. If $A_\bullet$ is a subcomplex of $C_\bullet$ then there is a long exact sequence

$$\ldots \longrightarrow H_n(A) \longrightarrow H_n(C) \longrightarrow H_n(C/A) \xrightarrow{\delta} H_{n-1}(A) \longrightarrow \ldots$$

Lemma 6.10. Let X be a topological space and $X' \subset X$ a subspace of X. The inclusion map defines an injective chain map $i_\# : C_\bullet(X') \longrightarrow C_\bullet(X)$.

Proof. Suppose $c = \sum m_i \sigma_j$ is a chain in X'.

$$i_\#(c) = \sum m_j \left(i \circ \sigma_j\right).$$

The maps $i \circ \sigma_i$ are all distinct since the σ_i are. $\square$

This means we can think of $C_\bullet(X')$ as a subcomplex of $C_\bullet(X)$ (under the isomorphism $C_\bullet(X') \cong \operatorname{Im} i_\#$, which we will suppress from the notation).

Notation. Write $C_\bullet(X, X')$ for the quotient $C_\bullet(X)/C_\bullet(X')$.

Theorem 6.11 (*LES in singular homology*). Let $X' \subset X$ be a subspace of a topological space X. Then there is a long exact sequence

$$\ldots \longrightarrow H_n(X') \longrightarrow H_n(X) \longrightarrow H_n(X, X') \longrightarrow H_{n-1}(X') \longrightarrow \ldots$$

Moreover, if $f : (X, X') \longrightarrow (Y, Y')$ is continuous, then the following diagram commutes.

$$\begin{tikzcd}
\ldots \arrow[r] & H_n(X') \arrow[r] \arrow[d, "H_n(f|_{X'})"] & H_n(X) \arrow[r] \arrow[d, "H_n(f)"] & H_n(X, X') \arrow[r, "\delta"] \arrow[d, "\text{induced}"] & H_{n-1}(X') \arrow[r] \arrow[d, "H_{n-1}(f|_{X'})"] & \ldots \\
\ldots \arrow[r] & H_n(Y') \arrow[r] & H_n(Y) \arrow[r] & H_n(Y, Y) \arrow[r, "\delta"] & H_{n-1}(Y') \arrow[r] & \ldots
\end{tikzcd}$$

This sequence will be responsible for most of our computations during the rest of the course.

The definition of $C_\bullet(X, X')$ is quite clunky to work with. We would like to not work with a quotient space since in practice it is not too useful. We give an alternative definition.

Definition 6.12. Let $X' \subset X$. Define

$$Z_n(X, X') := \left\{c \in C_n(X) : \ \partial c \in C_{n-1}(X')\right\}$$

be the set of relative n–cycles. If $X' = \emptyset$, then $X_n(X, \emptyset) = Z_n(X)$.

$$B_n := \left\{c \in C_n(X) : \ c - c' \in B_n(X) \text{ for some } c' \in C_n(X')\right\}.$$

Again, $B_n(X, \emptyset) = B_n(X)$.

Lemma 6.13.

$$H_n(X, X') \cong Z_n(X, X')/B_n(X, X') \quad \forall n \geq 0.$$

Proof. The boundary operator $\overline{\partial}$ of the quotient complex satisfies

$$\overline{\partial}\left(c + C_n(X')\right) = \partial c + C_{n-1}(X').$$

Thus $\ker \overline{\partial} = \{c : \partial c + C_{n-1}(X')$ is the zero element $\}$, so $\ker \overline{\partial} = Z_n(X, X')/C_n(X')$.

The image of $\overline{\partial}$ is $\{c + C_n(X') : \ c \in B_n(X')\} = B_n(X, X')/C_n(X')$.

Thus, by taking the quotient group, we get

$$H_n(X, X') = \frac{Z_n(X, X')/C_n(X')}{B_n(X, X')/C_n(X')} \cong Z_n(X, X')/B_n(X, X')$$

by the third theorem of isomorphisms of groups. □

This construction makes singular homology into a functor Top^2 $\longrightarrow$ Comp. It sends objects $(X, X') \mapsto H_n(X, X')$ and functions

$$f : (X, X') \longrightarrow (Y, Y') \longmapsto H_n(f) : H_n(X, X') \longrightarrow H_n(Y, Y')$$

induced by f.

Remark. The functor

$$\begin{array}{cccc} T: & \text{Top} & \longrightarrow & \text{Top}^2 \\ & X & \longmapsto & (X, \phi) \\ & f : X \longrightarrow Y & \longmapsto & f : (X, \emptyset) \longrightarrow (Y, \emptyset). \end{array}$$

is natural with respect to singular homology $H_\bullet^{\text{relative}} \circ T = H_\bullet^{\text{normal}}$.

Consider the case $X' = \{p\}$ is a single point. Then we have a long exact sequence

$$\ldots \longrightarrow H_n(p) \longrightarrow H_n(X) \longrightarrow H_n(X, p) \longrightarrow H_{n-1}(p) \longrightarrow \ldots$$

The homology of a 1–point space is

$$H_n(p) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0. \end{cases}$$

If $n \geq 2$ then we have

$$0 \longrightarrow H_n(X) \longrightarrow H_n(X, p) \longrightarrow 0,$$

i.e. $H_n(X) \cong H_n(X, p)$. Also,

$$0 \longrightarrow H_1(X) \longrightarrow H_1(X, p) \longrightarrow H_0(p) \longrightarrow H_0(X) \longrightarrow H_0(X, p) \longrightarrow 0.$$

Since $H_0(p) = \mathbb{Z}$ and $H_0(X)$ is always free abelian, we either have that $H_0(p) \longrightarrow H_0(X)$ is the zero map or it is injective. However, it is not the zero map since $\langle p \rangle$ is a generator of the path component containing p. Hence $H_1(X, p) \longrightarrow H_0(p)$ is surjective and thus $H_1(X, p)$ is isomorphic to $H_1(X)$. Finally, if X is path connected then $H_0(X, p) = 0$.

6.2 Reduced homology

We now introduce the notion of splitting a short exact sequence.

Definition 6.14. Suppose

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of abelian groups. We say the sequence splits if there exists $h : C \longrightarrow B$ such that $g \circ h = id_C$.

Lemma 6.15.

(i) Splitting is equivalent to asking that f has a one sided inverse, i.e. exists $k : B \longrightarrow A$ such that $k \circ f = id_A$.

(ii) If the sequence splits, the $B \cong A \oplus C$ but this is not canonical.

(iii) If C is free abelian then the sequence always splits.

Proof. We leave the proofs of (i) and (iii) as exercises and prove (ii). Let $h : C \longrightarrow B$ satisfy $g \circ h = id_C$. We claim that $B \cong \operatorname{Im} f \oplus \operatorname{Im} h$.

If $b \in B$ then $g(b)$ belongs to C and $b - h\left(g(b)\right)$ is in the kernel of g since $g\left(b - h(g(b))\right) = g(b) - g \circ h \circ g(b) = 0$. By exactness, $b - h\left(g(b)\right)$ is in the image of f. Therefore, $b = b - h(g(b)) + h(g(b))$ and $b - h(g(b)) \in \operatorname{Im} f$, $h(g(b)) \in \operatorname{Im} h$. However, this identification depends on the choice of map h. In other words, it may change depending on which map h is taken. Thus $B \cong A \oplus C$ is not natural and not canonical, as it depends on a choice. □

In general, H_n is only abelian, not free abelian. However, H_0 is free abelian. This means that we can at least split the tail end of the sequence, and gives rise to the following definition.

Definition 6.16. Let X be a topological space and $\{*\}$ a topological space with one point. Let j be the unique (continuous) map $j : X \longrightarrow \{*\}$. We define the reduced homology groups to be the kernel of $H(j)$.

$$\widetilde{H}_n(X) := \ker H_n(j) : H_n(X) \longrightarrow H_n(\{*\}).$$

Since this does not depend on choices, we can view reduced homology as a functor Top $\longrightarrow$ Comp. If $n \geq 1$, then $\widetilde{H}_n(X) \cong H_n(X)$. For $n = 0$ we have

$$0 \longrightarrow \widetilde{H}_0(X) \longrightarrow H_0(X) \xrightarrow{H_n(j)} H_0(\{*\}) \longrightarrow 0.$$

Here $H_0(\{*\}) \cong \mathbb{Z}$. Fix $p \in X$ and define $i_p : \{*\} \longrightarrow X$ such that $i_p(*) = p$. Thus

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z},$$

but the isomorphism is not canonical since it depends on the choice of $p \in X$.

Theorem 6.17 (*LES in reduced homology*). Let $X' \subset X$ be a subspace of a topological space X. Then there is a long exact sequence

$$\ldots \longrightarrow \widetilde{H}_n(X') \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(C_\bullet(X, X')) \longrightarrow \widetilde{H}_{n-1}(X') \longrightarrow \ldots$$

Moreover, if $f : (X, X') \longrightarrow (Y, Y')$ is continuous, the the following diagram commutes.

$$\begin{tikzcd}
\dots \arrow[r] & \widetilde{H}_n(X') \arrow[r] \arrow[d, "\widetilde{H}_n(f|_{X'})"] & \widetilde{H}_n(X) \arrow[r] \arrow[d, "\widetilde{H}_n(f)"] & \widetilde{H}_n(X,X') \arrow[r, "\delta"] \arrow[d, "\text{induced}"] & \widetilde{H}_{n-1}(X') \arrow[r] \arrow[d, "\widetilde{H}_{n-1}(f|_{X'})"] & \dots \\
\dots \arrow[r] & \widetilde{H}_n(Y') \arrow[r] & \widetilde{H}_n(Y) \arrow[r] & \widetilde{H}_n(Y,Y) \arrow[r, "\delta"] & \widetilde{H}_{n-1}(Y') \arrow[r] & \dots
\end{tikzcd}$$

The proof is left as an exercise. This theorem can be deduced from theorem 6.11. This sequence is usually easier to work with as it eliminates needing to compute the tail end of the long exact sequence. Another motivation for $\widetilde{H}$ is the following result.

Theorem 6.18. If (X, X') is a "well behaved" pair then

$$\widetilde{H}_n\left(X/X'\right) \cong H_n\left(X, X'\right).$$

Here, X/X' is the quotient topological space where we identify all points of X'. For example, if $X = B(0,1)$ and $X' = \partial X = \mathbb{S}^1$, then $B^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$. Applying the theorem,

$$\dots \longrightarrow H_i(\mathbb{S}^{n-1}) \longrightarrow H_i(B^n) \longrightarrow H_i(B^n, \mathbb{S}^{n-1}) \longrightarrow H_{i-1}(\mathbb{S}^{n-1}) \longrightarrow \dots$$

and $H_i(B^n, \mathbb{S}^{n-1}) \cong \widetilde{H}_i(\mathbb{S}^n)$. This allows us to compute $\widetilde{H}_0(\mathbb{S}^n)$ by induction.

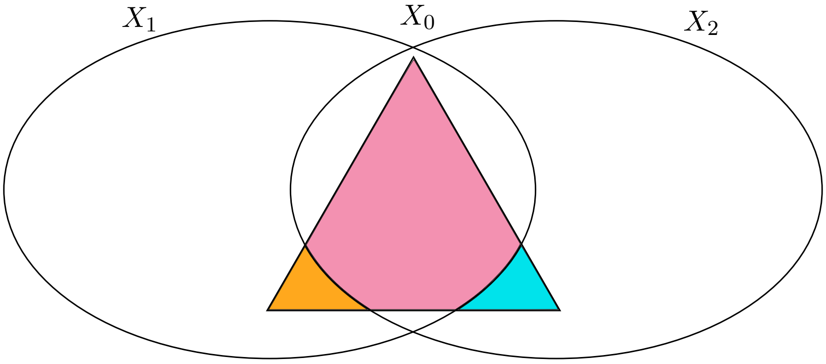
Theorem 6.19.

$$\widetilde{H}_\bullet(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \bullet = n \\ 0 \end{cases} \qquad n \geq 1$$

Case $n = 1$ follows from Hurewicz's theorem, and case $n \geq 1$ follows by induction. With this we now have proved all the results needed to justify the proof that we gave for Brouwer's fixed point theorem.

6.3 Barycentric subdivision

We now want to imitate the Seifert–van Kampen theorem in homology. The idea behind it is the same, although the execution is not as elegant.



We need a systematic way of "chopping up" a simplex into arbitrarily small other simplices. By doing this, we will be able to compute the homology of X in terms of the homology of X_0, X_1 and X_2. So our goal for now is precisely this.

Definition 6.20. Let $D \subset \mathbb{R}^n$ be a compact convex subset. Fix a point $p \in D$. If $\sigma : \Delta^n \longrightarrow D$ is a singular n–simplex, the cone of σ over p is the $(n+1)$–simplex $Q(p,\sigma)$

$$Q(p,\sigma)\left(s_0,...,s_{n+1}\right) = \begin{cases} p & s_0 = 1 \\ s_0 p + (1-s_0)\sigma\left(\frac{s_1}{1-s_0},...,\frac{s_{n+1}}{1-s_0}\right) & s_0 \neq 1. \end{cases}$$

We can think of $Q(p,\cdot) : C_n(D) \longrightarrow C_{n+1}(D)$. Notice that the second expression on the right hand side of the definition of Q is why we require convexity of D.

Lemma 6.21. Suppose $c = \sum m_i \sigma_i \in C_n(D)$. Then

$$\partial Q(p,c) = \begin{cases} c - Q(p,\partial c) & n \geq 1 \\ c - \left(\sum m_i\right) p & n = 0. \end{cases}$$

Proof. Assume $n \geq 1$. Consider $Q(p,\sigma) \circ \varepsilon_i$. This is

$$Q(p,\sigma) \circ \varepsilon_i(s_0,...,s_n) = \begin{cases} Q(p,\sigma)(0,s_1,...,s_n) & i = 0 \\ Q(p,\sigma)(s_0,...,0,...,s_n) & i \neq 0, \end{cases}$$

where the second term is

$$Q(p,\sigma)(s_0,...,0,...,s_n) = \begin{cases} Q(p,\sigma)(1,0,...,0) = p & s_0 = 1 \\ s_0 p + (1-s_0)\sigma\left(\frac{s_1}{1-s_0},...,0_{(i-1)},...,\frac{s_n}{1-s_0}\right) & s_0 \neq 1. \end{cases}$$

Therefore, we have

$$\begin{aligned} \partial Q(p,\sigma) = \sum (-1)^i Q(p,\sigma) \circ \varepsilon_i^{n+1} = \sigma + \sum_{i=0}^{n+1} (-1)^i Q(p, \sigma \circ \varepsilon_{i-1}^n) = \\ = \sigma - Q\left(p, \sum (-1)^i \sigma \circ \varepsilon_i^n\right) = \sigma - Q(p,\partial\sigma). \end{aligned}$$

□

Remark. Q is a chain homotopy for degrees $n \geq 1$ from id to 0. Thus, $H_n(D) = 0$ for all $n \geq 1$. This does not use the fact that H is a homotopy invariant.

Definition 6.22. Suppose $\sigma : \Delta^n \longrightarrow D$ is a n–simplex. We say σ is affine if

$$\begin{cases} \sigma\left(s_0,...,s_n\right) = \sum s_i \sigma(0,...,1_{(i)},...0) \\ \sigma\left(\sum s_i e_i\right) = \sum s_i \sigma(e_i). \end{cases}$$

If σ is affine then so is $\partial\sigma$ and $Q(p,\sigma)$. Also, $C_n^{\text{affine}} = A_n$ is a subcomplex.

Definition 6.23. We define the convex barycentric subdivision of an affine n–simplex

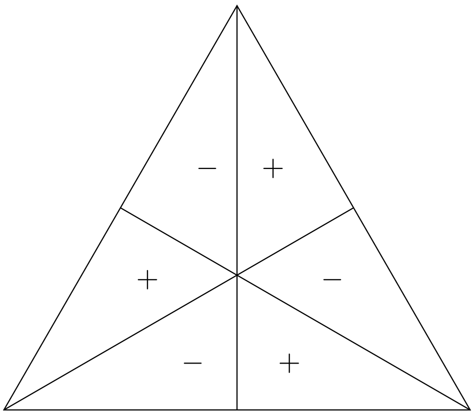


Figure 6.1: Subdivision of a 2–simplex.

as follows. Recall $b_n = \frac{1}{n+1}(e_0 + ... + e_n)$ is the barycentre of Δ^n. Now,

$$\begin{aligned} \mathrm{Sd}_n^{\mathrm{cv}} : A_n(D) &\longrightarrow A_n(D) \\ \sigma &\longmapsto \mathrm{Sd}_n^{\mathrm{cv}}(\sigma) = \begin{cases} \sigma & n = 0 \\ Q\left(\sigma(b_n), \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma)\right) & n \geq 1 \end{cases} \end{aligned}$$

This is how we define barycentric subdivision for simplices in topological spaces in general. Simplices need not be affine when the topological space is not a convex subset of the euclidean space. It is left as an exercise to check that $\mathrm{Sd}_2^{\mathrm{cv}}(\sigma)$ is as in figure 6.1. In general, suppose $\sigma : \Delta^n \longrightarrow X$ is a simplex

$$\Delta^n \xrightarrow{id} \Delta^n \xrightarrow{\sigma} X.$$

Then σ induces $\sigma_\# : C_n(\Delta^n) \longrightarrow C_n(X)$, for which $\sigma_\#(\ell_n) = \sigma$, where $\ell_n = id \in C_n(\Delta^n)$.

Definition 6.24. We define the barycentric subdivision

$$\begin{aligned} \mathrm{Sd}_n : C_n(X) &\longrightarrow C_n(X) \\ \sigma &\longmapsto \sigma_\#\left(\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n)\right). \end{aligned}$$

It is left as an exercise to check that if $\sigma \in A_n(D)$, then $\mathrm{Sd}_n(\sigma) = \mathrm{Sd}_n^{\mathrm{cv}}(\sigma)$.

Theorem 6.25. Barycentric subdivision induces a natural chain map

$$\mathrm{Sd}_\bullet : C_\bullet(X) \longrightarrow C_\bullet(X).$$

That is, if $f : X \longrightarrow Y$ is a continuous function, then the following diagram commutes

$$\begin{tikzcd} C_n(X) \arrow[r, "\mathrm{Sd}^X"] \arrow[d, "f_\#"'] & C_n(X) \arrow[d, "f_\#"] \\ C_n(Y) \arrow[r, "\mathrm{Sd}^Y"] & C_n(Y) \end{tikzcd}$$

Remark. We want to see that $H_\bullet(\mathrm{Sd}) = id$, but in order to prove this we need to prove the previous theorem first.

Proof of theorem 6.25. Using the notation from the theorem, let $\sigma \in C_n(X)$.

$$f_\# \operatorname{Sd}_n^X(\sigma) = f_\# \sigma_\# \left(\operatorname{Sd}_n^{\mathrm{cv}}(\ell_n)\right) = (f \circ \sigma)_\# \left(\operatorname{Sd}_n^{\mathrm{cv}}(\ell_n)\right) = \operatorname{Sd}_n^Y(f \circ \sigma) = \operatorname{Sd}_n^Y f_\#(\sigma).$$

Thus it suffices to show that $\operatorname{Sd}_\bullet^{\mathrm{cv}}$ is a chain map, since we already know $\sigma_\#$ is a chain map.

$$\begin{tikzcd}
C_n(\Delta^n) \arrow[r, "\operatorname{Sd}^{\mathrm{cv}}"] \arrow[d, "\sigma_\#"'] & C_n(\Delta^n) \arrow[d, "\sigma_\#"] \\
C_n(X) \arrow[r, "\operatorname{Sd}"] & C_n(X)
\end{tikzcd}$$

We prove $\partial_n \circ \operatorname{Sd}_n^{\mathrm{cv}} = \operatorname{Sd}_{n-1}^{\mathrm{cv}} \circ \partial_n$ by induction on n. For $n = 0$ we know it is true. For $n \geq 1$,

$$\begin{aligned}
\partial_n \operatorname{Sd}_n^{\mathrm{cv}}(\sigma) = \partial\left(Q(b_n, \operatorname{Sd}_{n-1}^{\mathrm{cv}}(\partial_n \sigma))\right) &= \operatorname{Sd}_n^{\mathrm{cv}}(\partial\sigma) - Q\left(b_n, \partial_{n-1} \operatorname{Sd}_n^{\mathrm{cv}} \partial\sigma\right) = \\
&= \operatorname{Sd}_{n-1}^{\mathrm{cv}}(\partial_n \sigma) - Q(b, \partial^2 \operatorname{Sd} \sigma) = 0
\end{aligned}$$

by the inductive hypothesis and since $\partial Q(p, c) = c - Q(p, \partial c)$. This shows that $\operatorname{Sd}_\bullet^{\mathrm{cv}}$ is a chain map. ☐

Theorem 6.26.

$$H\left(\operatorname{Sd}_\bullet\right) = id.$$

Remark. It is an immediate consequence of the Acyclic models theorem (which is similar to the homotopy axiom).

Proof. We construct a chain homotopy from $\operatorname{Sd}_\bullet$ to id. Explicitly, we want to find

$$P_n : C_n(X) \longrightarrow C_{n+1}(X)$$

such that

$$\partial P_n + P_{n-1}\partial = id - sd_n \tag{6.1}$$

Step 1. We find $P_n^{\mathrm{cv}} : A_n(D) \longrightarrow A_{n+1}(D)$ such that (6.1) holds for $\operatorname{Sd}^{\mathrm{cv}}$.

$$P_n^{\mathrm{cv}}(\sigma) = \begin{cases} 0 & n = 0 \\ Q\left(\sigma(b_n), \sigma - \operatorname{Sd}_n^{\mathrm{cv}}(\sigma) - P_{n-1}^{\mathrm{cv}}(\partial\sigma)\right) & n \neq 0. \end{cases}$$

The proof to show that (6.1) holds for P_n^{cv} is by induction over n.

$$\begin{aligned}
\partial\left(\sigma - \operatorname{Sd}_n^{\mathrm{cv}}(\sigma) - P_n^{\mathrm{cv}}(\partial\sigma)\right) = \partial\sigma - \partial \operatorname{Sd}_n(\sigma) - (id - \operatorname{Sd}_{n-1} - P_{n-2}\partial)(\partial\sigma) &= \\
= -\partial \operatorname{Sd}(\sigma) + \operatorname{Sd}(\partial\sigma) = 0.&
\end{aligned}$$

We now compute

$$\partial P_n^{\mathrm{cv}}(\sigma) = \partial Q(\ldots) = \sigma - \operatorname{Sd}(\sigma) - P\partial\sigma - Q(\cdot, \partial\ldots)$$

where the second argument of Q in the last expression is equal to 0 as we already showed.

Step 2. Define

$$\begin{aligned}
P_n : C_n(X) &\longrightarrow C_{n+1}(X) \\
\sigma &\longmapsto \sigma_\#\left(P_n^{\mathrm{cv}}(\ell_n)\right).
\end{aligned}$$

Then $P_\bullet$ is natural

$$\begin{tikzcd}
C_n(X) \arrow[r, "P_n^X"] \arrow[d, "f_\#"'] & C_{n+1}(X) \arrow[d, "f_\#"] \\
C_n(Y) \arrow[r, "P_n^Y"] & C_{n+1}(Y)
\end{tikzcd}$$

by the same argument as before.

Applying this with $X = \Delta, f = \sigma, Y = X$ and using step 1 shows that P satisfies (6.1). ☐

Definition 6.27. Let $\mathcal{U}$ be an open cover of X. Define a subcomplex $C_\bullet^{\mathcal{U}}(X) \subset C_\bullet(X)$ to be those $\sigma \in C_\bullet(X)$ such that $\exists U \in \mathcal{U}$ for which $\operatorname{Im}\sigma \subset U$. Let $H_\bullet^{\mathcal{U}}(X)$ denote its homology.

Theorem 6.28. The inclusion $i_\bullet : C_\bullet^{\mathcal{U}}(X) \longrightarrow C_\bullet(X)$ induces an isomorphism on homology $H_\bullet^{\mathcal{U}}(X) \cong H_\bullet(X)$.

The idea for the proof is to start with $\sigma \in C_\bullet(X)$ and then $\mathrm{Sd}^M(c) \in C_\bullet^{\mathcal{U}}(X)$ for some large M. Since $H(\mathrm{Sd}^M) = id^M = id$, this shows that the homology does not change. We will see the complete proof later, since some previous results are needed.

Lemma 6.29. Suppose $\sigma \in A_n(D)$. Write $z_i = \sigma(e_i)$. If $b = \frac{1}{n+1}(z_0, ..., z_n)$ is the barycentre, then for all $x, y \in \operatorname{Im}\sigma$ we have

$$\begin{aligned} |x - y| &\le \sup |z_i - y|, \\ \operatorname{diam}(\operatorname{Im}\sigma) &\le \sup |z_i - z_j|, \end{aligned}$$

and

$$|b - z_i| \le \frac{n}{n+1}\operatorname{diam}(\operatorname{Im}\sigma).$$

Definition 6.30. If $c \in A_n(D)$, we define

$$\operatorname{mesh} c = \max_{\sigma \in c} \operatorname{diam}(\operatorname{Im}\sigma).$$

Corollary 6.31.

$$\operatorname{mesh} \mathrm{Sd}_n^{\mathrm{cv}}(\sigma) \le \frac{n}{n+1}\operatorname{diam}(\operatorname{Im}\sigma).$$

The following corollary will allow us to subdivide a simplex enough times so that each one of the subdivisions is completely contained in an open set of the covering $\mathcal{U}$. This means that we will be able to translate the Seifert–van Kampen theorem into homology.

Corollary 6.32. If $\sigma \in C_n(X)$, then there exists M such that $\mathrm{Sd}^M(\sigma) \in C_n^{\mathcal{U}}(X)$.

Proof. By point–set topology, there exists $\delta > 0$ such that any open set V of Δ^n with diam $< \delta$ is contained in one of the sets of the open cover $\{\sigma^{-1}(U) : U \in \mathcal{U}\}$. Choose M such that

$\left(\frac{n}{n+1}\right)^M < \delta.$ □

Proof of theorem 6.28. We show $H_\bullet(i) : H_\bullet^{\mathcal{U}}(X) \longrightarrow H_\bullet(X)$ is injective (here $i = i_\bullet$).

Suppose $c \in C_n^{\mathcal{U}}(X)$ lies in the kernel of $H_n(i)$. This means $c = i_\# c$ is a boundary in $C_n(X)$, i.e. there exists $a \in C_{n+1}(X)$ such that $\partial a = c$. Using the lemma and the fact that a is a finite sum $\sum m_i \sigma_i$, we find $k \in \mathbb{N}$ such that $\operatorname{Sd}^k(a) \in C_{n+1}^{\mathcal{U}}(X)$. We proved that $\exists P$ such that

$$\partial P - P\partial = \operatorname{Sd} - id,$$

so by induction, if

$$p^{(k)} = P\left(id + \operatorname{Sd} + \ldots + \operatorname{Sd}^{k-1}\right)$$

then

$$\partial P^{(k)} + P^{(k)}\partial = \operatorname{Sd}^k - id,$$

and we have

$$\operatorname{Sd}^k(a) - a = \partial P^{(k)}(a) + P^{(k)}\partial(a) = \partial P^{(k)}(a) + P^{(k)}(c),$$

therefore, taking ∂ on both sides,

$$\partial \operatorname{Sd}^k(a) - c = \partial^2 P^{(k)}(a) + \partial P^{(k)}(c) = \partial P^{(k)}(c).$$

Thus

$$c = \partial\left[\operatorname{Sd}^{(k)}(a) - P^{(}k)(c)\right]$$

and so $c \in B_n^{\mathcal{U}}(X)$, $\langle c \rangle = 0$, since $\operatorname{Sd}^k(a), P^{(k)}(x) \in C_{n+1}^{\mathcal{U}}(X)$.

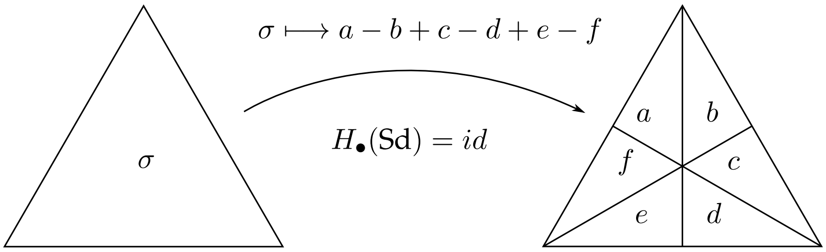
Now we prove surjectivity. Suppose $b \in Z_n(X)$ with homology class $\langle b \rangle \in H_n(X)$. We need to find a chain $d \in Z_n^{\mathcal{U}}(X)$ such that $H_n(i)\,\langle d \rangle = \langle d \rangle = \langle b \rangle$.

$$\partial P^{(k)}(b) + P^{(k)}(b) = \operatorname{Sd}^k(b) - b.$$

Sd is a chain map, and therefore $\operatorname{Sd}^k(b)$ is also a cycle. Thus

$$\left\langle \operatorname{Sd}^k b \right\rangle = \langle b \rangle$$

□



We have now finished the technical results we need in order to subdivide simplices into smaller ones without changing the homology, and we will now deal with the analogue for homology of the Seifert–van Kampen theorem, the Mayer–Vietoris theorem.

6.4 The Mayer–Vietoris theorem

Definition 6.33. Let $\mathcal{U}$ be an open cover of X and $X' \subset X$ a subset. Let $\mathcal{U} \cap X' = \{U \cap X' : U \in \mathcal{U}\}$. Then

$$C_\bullet^{\mathcal{U}}(X, X') = C_\bullet^{\mathcal{U}}(X)/C_\bullet^{\mathcal{U}\cap X'}(X').$$

Theorem 6.34. The inclusion $C_\bullet^{\mathcal{U}}(X, X') \longrightarrow C_\bullet(X, X')$ induces an isomorphism on homology.

Proof. The statement comes "for free" since we already know the absolute version. Consider the following chain complexes with exact rows

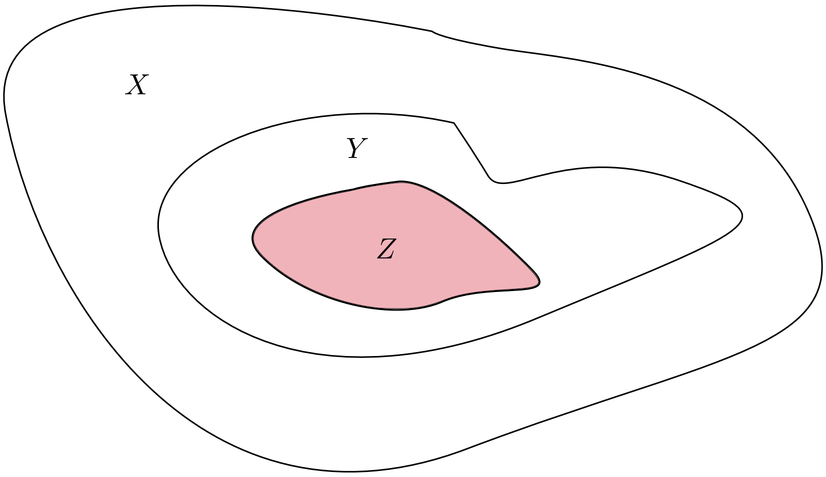
$$\begin{tikzcd}
0 \arrow[r] & C_\bullet^{\mathcal{U}\cap X'}(X') \arrow[r] \arrow[d] & C_\bullet^{\mathcal{U}}(X) \arrow[r] \arrow[d] & C_\bullet^{\mathcal{U}}(X, X') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_\bullet(X') \arrow[r] & C_\bullet(X) \arrow[r] & C_\bullet(X, X') \arrow[r] & 0
\end{tikzcd}$$

Apply long exact sequences to get

$$\begin{tikzcd}
H_n^{\mathcal{U}\cap X'}(X') \arrow[r] \arrow[d, "\cong"] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\cong"] & H_n^{\mathcal{U}}(X, X') \arrow[r] \arrow[d, "\cong"] & H_{n-1}^{\mathcal{U}\cap X'}(X') \arrow[r] \arrow[d, "\cong"] & H_{n-1}^{\mathcal{U}}(X) \arrow[d, "\cong"] \\
H_n(X') \arrow[r] & H_n(X) \arrow[r] & H_n(X, X') \arrow[r] & H_{n-1}(X') \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

where $H_n^{\mathcal{U}}(X, X') \cong H_n(X, X')$ is given by the Five lemma. □

Theorem 6.35 (*Excision*). Let X be a topological space and Z, Y subspaces of X such that $\overline{Z} \subset \operatorname{Int} Y \subset X$. Then $H_\bullet(X, Y) \cong H_\bullet(X \setminus Z, Y \setminus Z)$.



This theorem is in fact equivalent to the following, and that version is the one we will prove. The equivalence between both statements is left as an easy exercise.

Theorem 6.36. Let $X = X_1 \cup X_2$, where $X_1, X_2 \subset X$ are open subspaces. Set $X_0 = X_1 \cap X_2$. Then $H_\bullet(X_1, X_0) = H_\bullet(X, X_2)$.

Proof. Let $\mathcal{U}$ be an open cover $\{X_1, X_2\}$. Then $H_\bullet^{\mathcal{U}} = H_\bullet$ by theorem 6.28. By definition,

$$C_\bullet^{\mathcal{U}}(X) = C_\bullet(X_1) + C_\bullet(X_2).$$

Now consider the following short exact sequences

$$0 \longrightarrow C_\bullet(X_1) + C_\bullet(X_2) \longrightarrow C_\bullet(X) \longrightarrow \frac{C_\bullet(X)}{C_\bullet(X_1)+C_\bullet(X_2)} \longrightarrow 0$$

By the theorem 6.28, i induces an isomorphism in the long exact sequence.

$$A_n \xrightarrow{f} B_n \longrightarrow C_n \longrightarrow A_{n-1} \xrightarrow{f} B_{n-1} \longrightarrow C_{n-1}$$

If every third map is an isomorphism, then by exactness, every third group $C_\bullet$ is 0. Applying this to the previous sequence,

$$0 \longrightarrow \frac{C_\bullet(X_1)+C_\bullet(X_2)}{C_\bullet(X_2)} \longrightarrow \frac{C_\bullet(X)}{C_\bullet(X_2)} \longrightarrow \frac{C_\bullet(X)}{C_\bullet(X_1)+C_\bullet(X_2)} \longrightarrow 0$$

we find that there is an isomorphism of chain complexes

$$\frac{C_\bullet(X_1) + C_\bullet(X_2)}{C_\bullet(X_2)} \cong \frac{C_\bullet(X)}{C_\bullet(X_2)}$$

and by the second isomorphism theorem of groups,

$$\frac{C_\bullet(X_1)}{C_\bullet(X_0)} \cong \frac{C_\bullet(X_1) + C_\bullet(X_2)}{C_\bullet(X_2)}.$$

Putting this together,

$$H_\bullet\left(\frac{C_\bullet(X)}{C_\bullet(X_2)}\right) \cong H_\bullet\left(\frac{C_\bullet(X_1)}{C_\bullet(X_0)}\right)$$

or equivalently,

$$H_\bullet(X, X_2) \cong H_\bullet(X_1, X_0).$$

□

Theorem 6.37 (*Mayer–Vietoris*). Let $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$ so that the following diagram commutes

$$\begin{tikzcd}
 & X_0 \arrow[dl, hook, "i_1"'] \arrow[dr, hook, "i_2"] & \\
X_1 \arrow[dr, hook, "j_1"'] & & X_2 \arrow[dl, hook', "j_2"] \\
 & X &
\end{tikzcd}$$

There is a long exact sequence of the form

$$\dots \longrightarrow H_n(X_0) \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow H_{n-1}(X_0) \longrightarrow \dots$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
(X_0, \emptyset) \arrow[r] \arrow[d, "i_2"] & (X_1, \emptyset) \arrow[r] \arrow[d, "j_1"] & (X_1, X_0) \arrow[d, "h"] \\
(X_2, \emptyset) \arrow[r] & (X, \emptyset) \arrow[r] & (X, X_2)
\end{tikzcd}$$

Apply naturality and long exact sequences of a triple

$$\begin{tikzcd}
H_n(X_0) \arrow[r, "H_n(i_1)"] \arrow[d, "H_n(i_2)"] & H_n(X_1) \arrow[r] \arrow[d] & H_n(X_1, X_0) \arrow[r, "\delta"] \arrow[d, "H_n(h)"] & H_{n-1}(X_0) \arrow[d, "H_n(i_2)"] \\
H_n(X_2) \arrow[r, "H_n(j_2)"] & H_n(X) \arrow[r] & H_n(X, X_2) \arrow[r, "\delta"] & H_{n-1}(X_2)
\end{tikzcd}$$

Excision implies that $H_n(h)$ is an isomorphism. The following lemma yields the result. □

Lemma 6.38 (*Barret–Whitehead*). Given the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "g"] & B_n \arrow[r] \arrow[d, "k"] & C_n \arrow[r] \arrow[d, "\cong"] & A_{n-1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & D_n \arrow[r, "l"] & E_n \arrow[r] & F_n \arrow[r] & G_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

the following sequence is exact

$$\dots \longrightarrow A_n \xrightarrow{(f,g)} B_n \oplus D_n \xrightarrow{k-l} E_n \longrightarrow A_{n-1} \longrightarrow \dots$$

The proof for this lemma is quite lengthy, but is done by diagram chasing.

Corollary 6.39.

$$\widetilde{H}_i(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$

Proof. The previous proof we gave used $H(X, X') \cong \widetilde{H}(X/X')$. Now, let $\mathbb{S}^n = X_1 \cup X_2$ where X_1 consists of the sphere minus the north pole, and X_2 is the sphere minus the south pole. Then X_1 and X_2 are both contractible and $X_1 \cap X_2$ contracts onto the equator $\mathbb{S}^{n-1}$. Using the Mayer–Vietoris theorem, we get

$$\widetilde{H}_i(X_0) \longrightarrow \widetilde{H}_i(X_1) \oplus \widetilde{H}_i(X_2) \longrightarrow \widetilde{H}_i(X) \longrightarrow \widetilde{H}_{i-1}(X_0)$$

and since $\widetilde{H}_i(X_1) = \widetilde{H}_i(X_2) = 0$, we have

$$\widetilde{H}_i(\mathbb{S}^n) \cong \widetilde{H}_{i-1}(\mathbb{S}^{n-1}) \cong \widetilde{H}_{i-n}(\mathbb{S}^0) \cong \widetilde{H}_{i-n}(2\text{ pts}).$$

□

6.5 The degree

Suppose $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is continuous. Apply H_n to get

$$\begin{array}{ccc} H_n(f) : H_n(\mathbb{S}^n) & \longrightarrow & H_n(\mathbb{S}^n) \\ \mathbb{Z} & \longmapsto & \mathbb{Z} \end{array} .$$

This is necessarily multiplication by an integer, $H_n(f)\langle c\rangle = d\langle c\rangle$ for some $d \in \mathbb{Z}$, $\langle c\rangle$ a generator.

Definition 6.40. The degree of f is defined as $\deg f = d$.

Remark.

(i) In the case $n = 1$, we already defined the degree of a loop $u : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ and $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ with $[u] \mapsto \deg u$. Both definitions agree, which can be seen by Hurewicz's theorem.

(ii) The degree of $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is a higher dimensional version of the winding number.

(iii) If M is a closed, orientable topological manifold of dimension n, then $H_n(M) \cong \mathbb{Z}$ and we can define the degree of $f : M \longrightarrow M$. Recall that in differential geometry, a closed manifold is a compact manifold without boundary.

Theorem 6.41 (*Hairy ball theorem*). A tangential vector field on $\mathbb{S}^{2n}$ must vanish in at least one point.

There is a rather interesting meteorological interpretation of this theorem, that states that there exists at least a point on the surface of the Earth where there is no wind. The proof for this theorem will be given later.

Proposition 6.42. Let $n \geq 1$ and $f, g : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be two continuous maps. Then

(i) $\deg(g \circ f) = \deg g \deg f$.

(ii) $\deg(id) = 1$.

(iii) $\deg g = 0$ every time g is constant.

(iv) if $f \simeq g$ then $\deg f = \deg g$.

(v) if f is a homotopy equivalence, then $\deg f = \pm 1$.

These properties follow from the fact that H_n is a functor. To prove property (iii), it is useful to think of f as a map $\mathbb{S}^n \longrightarrow \mathbb{S}^n$ that can be factored through $\{*\}$.

Proposition 6.43. Let $n \geq 1$ and $A \in \mathbf{O}(n+1)$ be an orthogonal linear transformation. Then $\deg A|_{\mathbb{S}^n} = \det A$.

Proof. Since det is a continuous function on $\mathbf{O}(n+1)$ which takes values on $\{-1,1\}$, $\mathbf{O}(n+1)$ has two connected components, namely $\det^{-1}(\{1\})$ and $\det^{-1}(\{-1\})$. By homotopy invariances we need only check the result of one A in each of the two components, and since the identity map I_{n+1} induces a degree 1 map over $\mathbb{S}^n$, it is suffices to check the result for a map A with $\det A = -1$. Let us take A to be the reflection in a hyperplane $H \subset \mathbb{R}^{n+1}$. This divides $\mathbb{S}^n$ into two hemispheres that are the image of each other by A. Then $f = A|_{\mathbb{S}^n}$ induces a reflection f' of $\mathbb{S}^n$ over $H \cap \mathbb{S}^n \cong \mathbb{S}^{n-1}$. Now applying the Mayer–Vietoris theorem and using naturality, we obtain the commutative diagram

$$\begin{tikzcd}
H_n(\mathbb{S}^n) \arrow[r, "D"] \arrow[d, "H_n(f)"'] & H_{n-1}(\mathbb{S}^{n-1}) \arrow[d, "H_{n-1}(f')"] \\
H_n(\mathbb{S}^n) \arrow[r, "D"] & H_{n-1}(\mathbb{S}^{n-1})
\end{tikzcd}$$

where the maps D are the connecting maps given by the Mayer–Vietoris theorem, which are isomorphisms. Moreover, they are the same isomorphism. Thus we see that $\deg f = \deg f'$, and by construction it suffices to prove the result for $n = 1$.

Write $\mathbb{S}^1$ as the union of two open intervals A and B which contract onto the two given hemispheres preserved by the reflection. Then $A \cap B$ is homotopy equivalent to $\{p, q\}$ (which is $\mathbb{S}^0$). By applying the Mayer–Vietoris theorem again we see that $H_1(\mathbb{S}^1) \cong \ker j$, where

$$0 \longrightarrow H_1(\mathbb{S}^1) \longrightarrow H_0(\mathbb{S}^0) \xrightarrow{j} H_0(A) \oplus H_0(B).$$

Take $\langle p \rangle$ and $\langle q \rangle$ to be generators of $H_0(\mathbb{S}^0)$. Then both generate $H_0(A)$ and $H_0(B)$ and thus the map j is given by $(u, v) \longmapsto (u+v, u+v)$. In particular, its kernel is generated by $\langle p \rangle - \langle q \rangle$. Since the reflection f interchanges p and q, this shows that $\deg f = -1$. □

Corollary 6.44. Let $n \geq 1$. The antipodal map $a : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ given by $a(x) = -x$ has degree $(-1)^{n+1}$.

Proof. Note that a is the composition of $n+1$ reflections in the coordinate axes of $\mathbb{R}^{n+1}$. By the proposition, these are all of degree -1, and by the properties of the degree, a has degree the product of all the degrees, so $\deg a = \prod_{j=1}^{n+1}(-1) = (-1)^{n+1}$. □

We can now prove theorem 6.41. For this, it is useful to note that every vector field can be identified with a continuous map $v : \mathbb{S}^n \longrightarrow \mathbb{R}^{n+1}$ such that $x \cdot v(x) = 0$. Visually, we can think of $v(x)$ as a "hair" on $x \in \mathbb{S}^n$ which is tangential to the sphere. Then v is a vector field on $\mathbb{S}^n$. What the theorem states is that if $n = 2m$, then there will always be a point x where either $v(x) = 0$ (that is, there is no hair on x), or there will always be a tuft at x.

Proof of theorem 6.41. Let $n = 2m - 1$. Define

$$\begin{array}{rccc} v: & \mathbb{R}^{2m} & \longrightarrow & \mathbb{R}^{2m} \\ & (x_1, y_1, \ldots, x_m, y_m) & \longmapsto & (-y_1, x_1, \ldots, -y_m, x_m). \end{array}$$

Then the restriction $v|_{\mathbb{S}^n}$ is a nowhere vanishing vector field. We have just proved that if n is odd, then there exists a nowhere vanishing vector field on $\mathbb{S}^n$. Thus, theorem 6.41 is really an equivalence.

Suppose v is a nowhere vanishing vector field. Let

$$\begin{aligned} w : \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ x &\longmapsto \tfrac{v(x)}{|v(x)|}. \end{aligned}$$

Now define

$$\begin{aligned} W : \mathbb{S}^n \times I &\longrightarrow \mathbb{S}^n \\ (x,t) &\longmapsto (\cos \pi t)x + (\sin \pi t)w(x), \end{aligned}$$

a vector field on $\mathbb{S}^n$. Then $W(x,0) = x$ and $W(x,1) = a(x)$, where a is the antipodal map. Thus W is a homotopy and $\deg a = \deg id = 1$. Therefore, by the previous corollary, n must be odd. □

Definition 6.45. A continuous map $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is said to be odd if $f(-x) = -f(x)$, $x \in \mathbb{S}^n$.

Theorem 6.46. Every odd map has odd degree.

Thanks to this theorem one can prove two classical results, namely the Borsuk–Ulam theorem and the Lusternik–Schnirelmann theorem. To prove it, we will need some properties of odd functions first.

Proposition 6.47. Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be an odd map. Then there exists an odd map $f' : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ such that $f'(\mathbb{S}^i) \subset \mathbb{S}^i$ for every $i = 0, ..., n$ and a homotopy $F : f \simeq f'$ such that $f_t(x) := F(x,t)$ is an odd map for every $t \in I$.

We will not see the proof for this result since it requires some tools we have not yet developed. For now, let us believe that the proposition is true. Then this implies that we can now prove theorem 6.46 via the following result.

Proposition 6.48. Let $n \geq 1$ and $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be an odd map such that $f(\mathbb{S}^i) \subset \mathbb{S}^i$ for every $i = 0, 1, ..., n$. Then $\deg f$ has the same parity as $\deg f|_{\mathbb{S}^{n-1}}$.

Note that since the only odd maps $\mathbb{S}^0 \longrightarrow \mathbb{S}^0$ are the identity and the antipodal maps (which both have odd degree), theorem 6.46 follows easily from the proposition.

Proof of proposition 6.48. Consider the following commutative diagram

$$\begin{tikzcd}
 & & \widetilde{H}_n(\mathbb{S}^n) \arrow[dll, "l_+"'] \arrow[drr, "l_-"] \arrow[dd, "h"] & & \\
H_n(\mathbb{S}^n, B^n_+) & & & & H_n(\mathbb{S}^n, B^n_-) \\
 & & H_n(\mathbb{S}^n, \mathbb{S}^{n-1}) \arrow[ull, "j_+"] \arrow[urr, "j_-"'] \arrow[dd, "\delta"] & & \\
H_n(B^n_-, \mathbb{S}^{n-1}) \arrow[uu, "k_+"] \arrow[urr, "i_-"'] \arrow[drr, "\delta_+"'] & & & & H_n(B^n_+, \mathbb{S}^{n-1}) \arrow[uu, "k_-"'] \arrow[ull, "i_+"] \arrow[dll, "\delta_-"] \\
 & & \widetilde{H}_{n-1}(\mathbb{S}^{n-1}) & &
\end{tikzcd}$$

We will use the hexagon lemma, stated below. We used reduced homology at the top and bottom just so that the proof still works for the case $n = 1$. The maps $\delta_\pm$ and δ are the ones that come from the long exact sequence for reduced homology, and all the other maps are the induced ones by the inclusions. Moreover, the groups that form the vertices of the hexagon (all but the middle one) are isomorphic to $\mathbb{Z}$. The maps $k_\pm, l_\pm$ and $\delta_\pm$ are all isomorphisms. Exactness holds at $H_n(\mathbb{S}^n, \mathbb{S}^{n-1})$ for all three diagonals, $\operatorname{Im} i_- = \ker j_-$, $\operatorname{Im} i_+ = \ker j_+$ and $\operatorname{Im} h = \ker \delta$. The map f induces maps

$$H_n(f) : \widetilde{H}_n(\mathbb{S}^n) \longrightarrow \widetilde{H}_n(\mathbb{S}^n), \quad H_{n-1}(f|_{\mathbb{S}^{n-1}}) : \widetilde{H}_{n-1}(\mathbb{S}^{n-1}) \longrightarrow \widetilde{H}_{n-1}(\mathbb{S}^{n-1})$$

and by assumption, a map $H_n(\mathbb{S}^n, \mathbb{S}^{n-1}) \longrightarrow H_n(\mathbb{S}^n, \mathbb{S}^{n-1})$. We will denote them all by φ. Similarly, we will denote by α all the maps on homology induced by the antipodal map a. Now fix a generator $\langle c \rangle \in \widetilde{H}_{n-1}(\mathbb{S}^{n-1})$. This uniquely defines generators $\langle c_\pm \rangle \in H_n(B^n_\mp, \mathbb{S}^{n-1})$ thanks to the equation $\delta_\pm(\langle c_\pm \rangle) = \langle c \rangle$. Also, if $\langle b_\pm \rangle := k_\pm(\langle c_\pm \rangle)$ then $\langle b_\pm \rangle$ are generators of $H_n(\mathbb{S}^n, B^n_\pm)$, and there exist $\langle a_\pm \rangle \in \widetilde{H}_n(\mathbb{S}^n)$ such that $l_\pm(\langle a_\pm \rangle) = \langle b_\pm \rangle$. Now we set $\langle u_\pm \rangle = i_\mp(\langle c_\pm \rangle)$. By diagram chasing we can deduce that $\{\langle u_+ \rangle, \langle u_- \rangle\}$ is a basis of $H_n(\mathbb{S}^n, \mathbb{S}^{n-1})$, and in particular

$$H_n(\mathbb{S}^n, \mathbb{S}^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Now let $d := \deg f$ and $d' = \deg f|_{\mathbb{S}^{n-1}}$. Then

$$\varphi(\langle a_+ \rangle) = d \langle a_+ \rangle, \quad \varphi(\langle c \rangle) = d' \langle c \rangle .$$

Since $\{\langle u_+ \rangle, \langle u_- \rangle\}$ is a basis, there exists integers p, q such that $\varphi(\langle u_+ \rangle) = p \langle u_+ \rangle + q \langle u_- \rangle$. Now we will show that

$$d = p - 1, \quad d' = p + q.$$

in order to complete the proof. Note that since $\delta(\langle u_\pm \rangle) = \langle c \rangle$ by commutativity,

$$\varphi(\langle c \rangle) = \varphi(\delta(\langle u_+ \rangle)) = \delta(\varphi(\langle u_+ \rangle)) = \delta(p \langle u_+ \rangle + q \langle u_- \rangle) = (p + q) \langle c \rangle ,$$

where we used naturality for $\varphi \circ \delta = \delta \circ \varphi$. Thus $d' = p + q$. Now consider α. By naturality, $\delta_- \circ \alpha = \alpha \circ \delta_+$, and we have $\alpha(\langle c \rangle) = (-1)^n \langle c \rangle$. Thus also $\alpha(\langle c_+ \rangle) = (-1)^n \langle c_- \rangle$ and $\alpha(\langle u_- \rangle) = (-1)^n \langle u_+ \rangle$. Next, since f is odd, $\varphi \circ \alpha = \alpha \circ \varphi$ and

$$\begin{aligned} \varphi(\langle u_- \rangle) &= (-1)^n \varphi\left(\alpha(\langle u_+ \rangle)\right) = (-1)^n \alpha\left(\varphi(\langle u_+ \rangle)\right) = \\ &= (-1)^n \alpha\left(p \langle u_+ \rangle + q \langle u_- \rangle\right) = p \langle u_- \rangle + q \langle u_+ \rangle . \end{aligned}$$

Now since $\operatorname{Im} h = \ker \delta$, the image of H must be generated by $\langle u_+ - u_- \rangle$. Therefore $h(\langle a_+ \rangle) = r \langle u_+ - u_- \rangle$ with $r = \pm 1$. In fact, we claim $r = 1$. Using the fact that $j_+(\langle u_+ \rangle) = l_+(\langle a_+ \rangle)$, we have

$$j_+(\langle u_+ \rangle) = l_+(\langle a_+ \rangle) = j_+\left(h(\langle a_+ \rangle)\right) = r j_+(\langle u_+ - u_- \rangle) = r j_+(\langle u_+ \rangle)$$

since $\langle u_+ \rangle$ is in $\operatorname{Im} i_+ = \ker j_+$. By definition of d,

$$\begin{aligned} d \langle u_+ - u_- \rangle = dh(\langle a_+ \rangle) = \varphi(h(\langle a_+ \rangle)) = \varphi(\langle u_+ - u_- \rangle) = \\ = \langle p u_+ + q u_- \rangle - \langle p u_- + q u_+ \rangle = (p-1) \langle u_+ - u_- \rangle . \end{aligned}$$

Hence $d = p - q$, which completes the proof. ☐

To finish this chapter, let us take a look at the lemma that we used for this proof.

Lemma 6.49 (*The hexagon lemma*). Suppose the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "l"] & \\
B_1 & & B_2 \\
& C \arrow[ul, "j_1"] \arrow[ur, "j_2"'] \arrow[dd, "h"] & \\
D_2 \arrow[uu, "k_1"] \arrow[ur, "i_2"'] \arrow[dr, "g_1"'] & & D_2 \arrow[uu, "k_2"'] \arrow[ul, "i_1"] \arrow[dl, "g_2"] \\
& E &
\end{tikzcd}$$

Assume k_1 and k_2 are isomorphisms and $\operatorname{Im} i_1 \subset \ker j_1$ and $\operatorname{Im} i_2 \subset \ker j_2$. Then $C \cong D_1 \oplus D_2 \cong B_1 \oplus B_2$ and $\operatorname{Im} i_1 = \ker j_1$ and $\operatorname{Im} i_2 = \ker j_2$. Also, if $\operatorname{Im} l \subset \ker h$, then

$$g_1 \circ k_1^{-1} \circ f_1 = g_2 \circ k_2^{-1} \circ f_2 .$$

The proof of the hexagon lemma consists in diagram chasing.

7 Colimits

We will now go back to category theory to formalise the notion of a pushout and extend it to diagrams with more general shapes. That will be our goal for this part, given a diagram of shape $\mathcal{J}$, define a "most efficient solution", which we will call a colimit. This abstract machinery that we are going to deal with will help us when proving Brouwer's invariance of domain theorem later.

Definition 7.1. Let $\mathcal{J}$ be a small category. Let $\mathcal{C}$ be another category. A diagram of shape $\mathcal{J}$ in $\mathcal{C}$ is a functor $T : \mathcal{J} \longrightarrow \mathcal{C}$. We call $\mathcal{J}$ an index category.

We now generalise the notion of a solution to a diagram. We will usually call α, β, γ the objects in the indexing category $\mathcal{J}$. Morphisms will be denoted by i, j.

Definition 7.2. Let $\mathcal{J}$ be an index category and $T : \mathcal{J} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$. A solution for T is an object C of $\mathcal{C}$ together with a family of morphisms $c_\alpha : T(\alpha) \longrightarrow C$ in $\mathcal{C}$ for each object $\alpha \in \operatorname{obj} \mathcal{J}$ such that if $i : \alpha \longrightarrow \beta$ is any morphism in $\mathcal{J}$ then the following commutes

$$\begin{tikzcd}
T(\alpha) \arrow[r, "c_\alpha"] \arrow[d, "T(i)"'] & C \\
T(\beta) \arrow[ur, "c_\beta"'] &
\end{tikzcd}$$

We will write $(C, \{c_\alpha\})$ to denote the solution.

Now let us generalise the notion of a pushout.

Definition 7.3. Let $\mathcal{J}$ be an index category and let $T : \mathcal{J} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$. A colimit is a solution $(L, \{l_\alpha\})$ that satisfies the following universal property. If $(C, \{c_\alpha\})$ is another solution, then there exists a unique morphism $u : L \longrightarrow C$ such that the following diagram commutes for every morphism $i : \alpha \longrightarrow \beta$ in $\mathcal{J}$:

$$\begin{tikzcd}
& T(\beta) \arrow[d, "l_\beta"] \arrow[ddr, "c_\beta"] & \\
T(\alpha) \arrow[ur, "T(i)"] \arrow[r, "l_\alpha"] \arrow[drr, "c_\alpha"'] & L \arrow[dr, dotted, "u"] & \\
& & C
\end{tikzcd}$$

Lemma 7.4. A colimit is unique up to ismomorphism if it exists.

Proof. Suppose $(L, \{l_\alpha\})$ and $(K, \{k_\alpha\})$ are two colimits. Then there exist unique maps f, g

$$\begin{tikzcd}
& T(\beta) \arrow[d, "l_\beta"] \arrow[ddr, "c_\beta"] & \\
T(\alpha) \arrow[ur, "T(i)"] \arrow[r, "l_\alpha"] \arrow[drr, "c_\alpha"'] & L \arrow[dr, dotted, "f"] & \\
& & C
\end{tikzcd}
\qquad
\begin{tikzcd}
& T(\beta) \arrow[d, "k_\beta"] \arrow[ddr, "c_\beta"] & \\
T(\alpha) \arrow[ur, "T(i)"] \arrow[r, "k_\alpha"] \arrow[drr, "c_\alpha"'] & K \arrow[dr, dotted, "g"] & \\
& & C
\end{tikzcd}$$

and splicing the diagrams together, we find that there is a unique map that coincides with $g \circ f$ by viewing L as a limit. Since the identity map works, by uniqueness we have $g \circ f = id_L$ and $f \circ g = id_K$, so f is an isomorphism. □

Some examples of colimits that we already know are pushouts and coproducts, the former being the colimit of

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet & \rightsquigarrow & A \arrow[r, "f"] \arrow[d, "g"'] & B \\
\bullet & & & C &
\end{tikzcd}$$

and the latter being the colimit of

$$\begin{tikzcd}
\bullet & \bullet & \rightsquigarrow & A & B
\end{tikzcd}$$

Lemma 7.5. Suppose $\mathcal{C}$ is Sets. Then coproducts exist and they are disjoint unions. In Groups, coproducts exist and they are the free products.

The proof for this lemma is left as an exercise, and it is carried out by unraveling the definitions. The categories we will use will always have coproducts.

Remark. Suppose $T : \mathcal{J} \longrightarrow \mathcal{C}$ is a diagram and suppose $S : \mathcal{C} \longrightarrow \mathcal{D}$ is another functor. Then $S \circ T : \mathcal{J} \longrightarrow \mathcal{D}$ is a diagram in $\mathcal{D}$. Assume the colimits exist: $\operatorname{colim} T$ in $\mathcal{C}$ and $\operatorname{colim} S \circ T$ in $\mathcal{D}$. We claim there is a natural morphism

$$u : \operatorname{colim} S \circ T \longrightarrow S(\operatorname{colim} T)$$

that is, $u \in \operatorname{Hom}_{\mathcal{D}}(\operatorname{colim} S \circ T, S(\operatorname{colim} T))$.

$$\begin{tikzcd}
T(\alpha) \arrow[d, "T(i)"'] \arrow[dr, "c_\alpha"] & & & ST(\alpha) \arrow[d, "ST(i)"'] \arrow[dr, "S(c_\alpha)"] & \\
T(\beta) \arrow[r, "c_\beta"] & \operatorname{colim} T & \overset{S}{\longmapsto} & ST(\beta) \arrow[r, "S(c_\beta)"] & S(\operatorname{colim} T)
\end{tikzcd}$$

7.1 Filtered colimits

We will now deal with filtered colimits, which are a special type of colimit that is easier to handle.

Definition 7.6. Let $\mathcal{J}$ be an index category with a nonempty class of objects. We say $\mathcal{J}$ is filtered if for all objects α, β in $\mathcal{J}$,

(i) there exists another object γ together with morphisms $\alpha \longrightarrow \gamma$, $\beta \longrightarrow \gamma$.

(ii) if

$$\begin{tikzcd} \alpha \arrow[r, leftrightarrow, "i", "j"'] & \beta \end{tikzcd}$$

are morphisms, then there exists another object γ and a morphism $k : \beta \longrightarrow \gamma$ such that

$$\begin{tikzcd} & \beta \arrow[dr, "k"] & \\ \alpha \arrow[ur, "i"'] \arrow[dr, "j"'] & & \gamma \\ & \beta \arrow[ur, "k"] & \end{tikzcd}$$

Definition 7.7. A directed set $(\Lambda, \preccurlyeq)$ where $\preccurlyeq$ is reflexive and transitive and such that $\forall \alpha, \beta \in \Lambda\ \exists \gamma$ such that $\alpha \preccurlyeq \gamma, \beta \preccurlyeq \gamma$.

For example, $(\mathbb{N}, \leq)$ is a directed set. If $(\Lambda, \preccurlyeq)$ is a directed set, one can create a filtered index category $\mathcal{J} = \mathcal{J}(\Lambda, \preccurlyeq)$ by setting obj $\mathcal{J} = \Lambda$ and if $\alpha, \beta \in$ obj $\mathcal{J}$, set

$$\mathrm{Hom}_{\mathcal{J}}(\alpha, \beta) = \begin{cases} i_{\alpha,\beta} & \alpha \preccurlyeq \beta \\ \emptyset & \text{otherwise} \end{cases}$$

This is filtered because $(\Lambda, \preccurlyeq)$ is filtered. A filtered colimit is a colimit on a filtered category. We will indicate that a colimit is a filtered colimit by writing $\underrightarrow{\mathrm{colim}}$. For example, given $(\mathbb{N}, \leq)$, define $\mathcal{J}$ whose objects are $\mathbb{N}$ and whose morphisms are

$$\mathrm{Hom}_{\mathcal{J}}(m, n) = \begin{cases} i_{m,n} & m \leq n \\ \emptyset & m > n \end{cases}$$

For each $n \in \mathbb{Z}$ we have an object $C_n = T(n)$ in $\mathcal{C}$ and $T(i_{m,n}) : C_m \longrightarrow C_n$ for $m \leq n$. Set $f_n : C_n \longrightarrow C_{n+1}$ to be $T(i_{n,n+1})$. Then, since T is a functor,

$$T(i_{m,n}) = f_{n-1} \circ \ldots \circ f_{m+1} \circ f_m : C_m \longrightarrow C_n.$$

In other words, T is equivalent to a collection $f_n : C_n \longrightarrow C_{n+1}$, $n \in \mathbb{N}$.

$$\begin{tikzcd} & C_{n+1} \arrow[dr, "l_{n+1}"] \arrow[ddrr, bend left] & & \\ C_n \arrow[ur, "f_n"] \arrow[rr, "l_n"'] \arrow[drrr, bend right] & & \underrightarrow{\mathrm{colim}}\, T & \\ & & & L \end{tikzcd}$$

Remark. In this, the colimit is often called "direct limit" and written $\varinjlim$. Thus direct limits are a special case of filtered colimits.

Proposition 7.8. Filtered colimits exist in Sets, Top, Groups, Ab, Comp, ...

Proof.

Sets

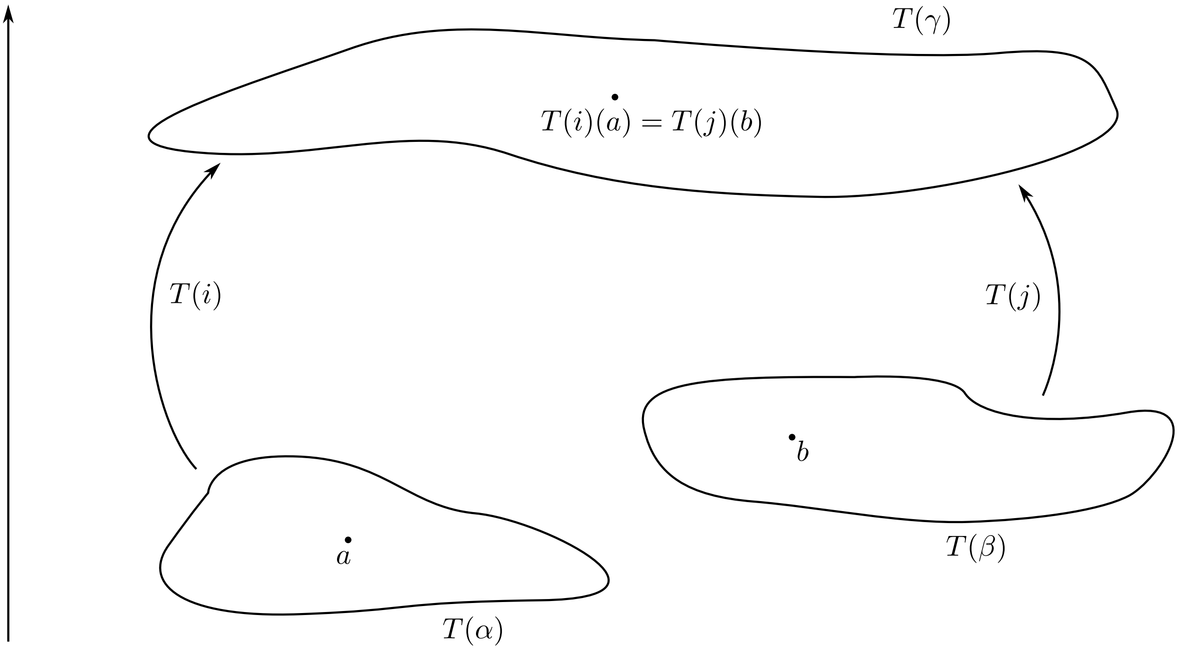
Suppose $\mathcal{J}$ is a filtered index category and $T : \mathcal{J} \longrightarrow$ Sets is a functor. First, form the disjoint union of all the sets $T(\alpha)$.

$$Z = \bigsqcup_{\alpha \in \text{obj}\,\mathcal{J}} T(\alpha).$$

This is a solution, but it is not the "most efficient" one as Z is too "big".

Define an equivalence relation in Z. We say $a \in T(\alpha)$ is equivalent to an element $b \in T(\beta)$ if there exists $\gamma \in \text{obj}\,\mathcal{J}$ and maps $i : \alpha \longrightarrow \gamma$, $j : \beta \longrightarrow \gamma$ such that

$$T(i)(a) = T(j)(b) \quad \text{as elements of } T(\gamma).$$



This is an equivalence relation as $\mathcal{J}$ is filtered and $Z/\sim$ is a filtered colimit

$$\begin{tikzcd} T(\alpha) \arrow[r, hook] & \bigsqcup T(\alpha) \arrow[r, "\text{pr}"] & Z/\sim \end{tikzcd}$$

Top

With Z as before, we get something that will function as a filtered colimit in Top once we give it a topology

$$\begin{tikzcd} T(\alpha) \arrow[r, hook] \arrow[rr, bend right, "l_\alpha"'] & \bigsqcup T(\alpha) \arrow[r, "\text{pr}"] & Z/\sim \end{tikzcd}$$

We need to equip $Z/\sim$ with a topology such that all the maps l_α are continuous. We do so by declaring that l_α are in fact continuous, that is, a set $U \subset X$ is open if and only if $l_\alpha^{-1}(U)$ is open in $T(\alpha)$ for all $\alpha \in \operatorname{obj}\mathcal{J}$. Recalling the forgetful functor

$$\begin{array}{rccc} \text{Forget}: & \text{Top} & \longrightarrow & \text{Sets} \\ & \mathcal{J} & \longmapsto & \text{Top} \\ & u : \varinjlim(\text{Forget}\circ T) & \longmapsto & \text{Forget}\left(\varinjlim T\right). \end{array}$$

In this case u is the identity map in $Z/\sim$.

Ab

Assume each $T(\alpha)$ is an abelian group. Then to get a filtered colimit in Ab we need only define an abelian group structure on $A = Z/\sim$ such that each $l_\alpha : T(\alpha) \longrightarrow A$ is a group homomorphism. Namely, for each $a \in T(\alpha), b \in T(\beta)$, choose γ and $i : \alpha \longrightarrow \gamma$, $j : \beta \longrightarrow \gamma$, define

$$[a] + [b] := [T(i)(a) + T(j)(b)].$$

Comp

Similarly, filtered colimits exist in Comp.

□

Now let $T : \mathcal{J} \longrightarrow$ Comp be a filtered diagram with colimit $\varinjlim T$ (a chain complex). The algebraic homology functor $\mathbb{H}$: Comp $\longrightarrow$ Comp induces a natural morphism for each T, namely

$$u : \varinjlim \mathbb{H}\circ T \longrightarrow \mathbb{H}(\varinjlim T).$$

Theorem 7.9. In Comp, the map u is an isomorphism.

Let $C_\bullet^\alpha$ be the chain complex $T(\alpha)$. What the theorem states is that

$$H_\bullet\left(\varinjlim C_\bullet^\alpha\right) = \varinjlim H_\bullet\left(C_\bullet^\alpha\right)$$

i.e. homology and colimits commute.

Proof. To prove u is surjective, let us take $z \in \varinjlim C_\bullet^\alpha$ is a cycle of degree n. Choose a representative $a \in C_n^\alpha$ of z. Now a need not be a cycle in $(C_\bullet^\alpha, \partial^\alpha)$. Since a eventually is a cycle, there exists $\beta \in \operatorname{obj}\mathcal{J}$ and a morphism $i : \alpha \longrightarrow \beta$ in $\mathcal{J}$ such that $T(i)(a)$ is a cycle in C_n^β, i.e. $\partial^\beta\left(T(i)(a)\right) = 0$. Thus $\langle T(i)(a)\rangle \in H_n(C_\bullet^\beta)$. Let $x \in \varinjlim\left(\mathbb{H}\circ T\right)$ be the element represented by $\langle T(i)(a)\rangle$. Then by definition $u(x) = \langle z\rangle$.

Injectivity requires a similar argument, and it is left as an exercise. □

7.2 Colimits in Top

Suppose we have a collection of embeddings

$$i_n : X_n \hookrightarrow X_{n+1} \quad n \in \mathbb{Z}.$$

(This is the same thing as a diagram on the filtered index category $\mathbb{N}$). Since i_n is an embedding, we can replace X_n with the homeomorphic space $i_n(X_n)$ and thus regard X_n as a subspace of X_{n+1} (with i_n the inclusion). The space $\varinjlim_n X_n$ is the union $X = \bigcup_{n\in\mathbb{N}} X_n$ with the topology for which a set C is closed in X if and only if $C \cap X_n$ is closed for all n.

The definition of the colimit gives a map $l_n : X_n \longrightarrow X$ (the inclusion). However, in general l_n might not be an embedding. The way to circumvent this is to take X_n either open or closed in X_{n+1} for every $n \in \mathbb{N}$. In this case l_n is an embedding.

Theorem 7.10. Assume $X = \bigcup X_n$ where $X_n \hookrightarrow X_{n+1}$ is either

(i) open for every $n \in \mathbb{Z}$.

(ii) closed for every $n \in \mathbb{Z}$ and that each X_n is a T_1 space.

Then $C_\bullet$ also commutes with the colimit

$$C_\bullet(X) = C_\bullet\left(\varinjlim X_n\right) = \varinjlim C_\bullet(X_n).$$

Corollary 7.11. Assume X is as before. Then

$$H_\bullet(X) = \varinjlim H_\bullet(X_n).$$

Here X carries the colimit topology, namely X is closed if and only if $C \cap X_n$ is closed for every n. This topology is not always that which one would expect. For example, if $X = \mathbb{S}^1$ viewed as a quotient and X_n is an interval, then $H_1(\mathbb{S}^1) = \mathbb{Z}$ but $\varinjlim H_1(X_n) = 0$.

Proof. Combine the previous theorem with theorem 7.9. □

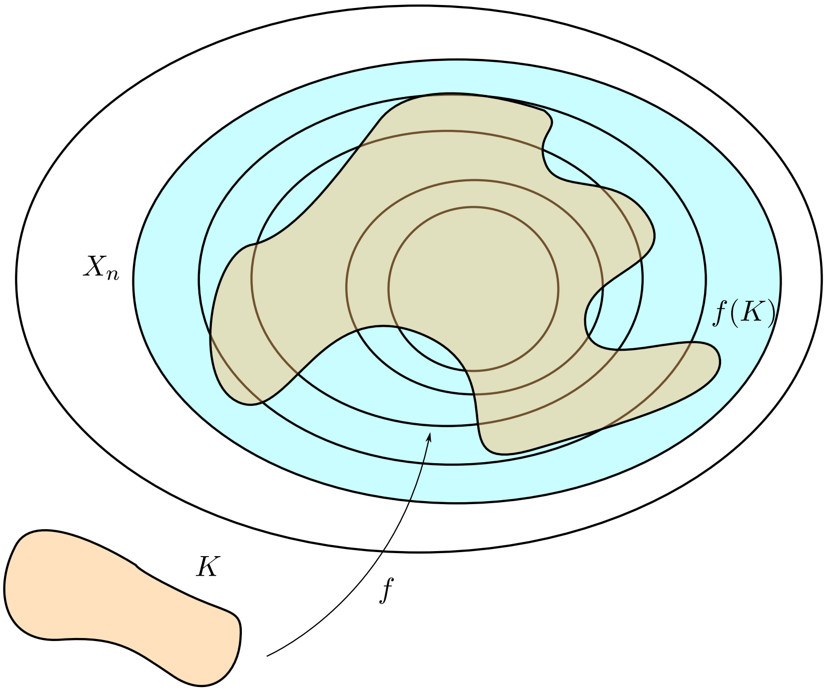
Recall that for a space to be T_1 it is required that its points are closed. This condition is strictly weaker that Hausdorff. Since T_1 is not restrictive enough and Haudorff is too strict, we establish a new condition called "weakly Hausdorff". A space is weakly Hausdorff if every time $f : K \longleftarrow X$ is a continuous map where K is compact Hausdorff, then $f(K)$ is closed in X.

Lemma 7.12. If $f : K \longrightarrow X$ is a continuous map from a compact space, then there exists n such that $f(K) \subset X_n$.

Proof. In case (i), the sets $\{X_n\}$ form an open cover of X. Since $f(K)$ is compact, there is a finite subcover of $\{X_n\}$ that cover $f(K)$, and thus there exists m such that $f(K) \subset X_m$.

In case (ii), argue by contradiction. There exists a sequence $x_n \in K$ such that $f(x_n) \notin X_n$ for all $n \in \mathbb{N}$. Set $S_m = \{f(x) : n \geq m\}$. Then $S_{m+1} \subset S_m$ and $\bigcap_m S_m = \emptyset$. Moreover, $S_m \cap X_k$ is a finite set for all m, k. Since each X_k is T_1, the finite set $S_m \cap X_k$ is closed in X_k. By definition of the colimit topology, S_m is closed in X for all m. Thus $Y_m = X \setminus S_m$ is open in X for each m. The Y_m's form an open cover of X and hence form a cover of $f(K)$. By construction, there is no finite subcover of the compact set $f(K)$. □

Proof of theorem 7.10. If $\sigma : \Delta^i \longrightarrow X$ is a singular i–simplex, then by the lemma, $\sigma(\Delta^i)$ is contained in some X_n and hence σ is a singular i–simplex in X_n. Thus by definition of the

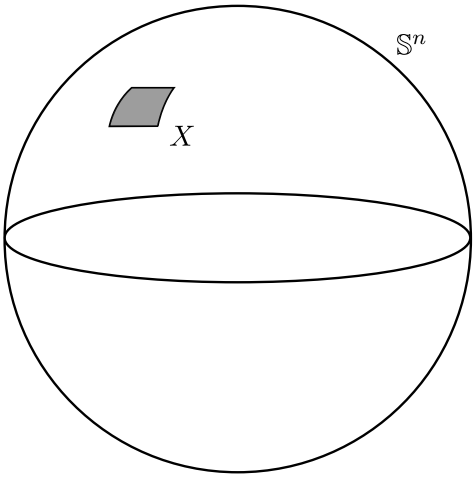


colimit in Comp,

$$C_\bullet(X) = \operatorname{colim} C_\bullet(X_n).$$

□

The following proposition is the key technical result to prove Brouwer's invariance of domain theorem.



Proposition 7.13. Let X be a subset of $\mathbb{S}^n$ which is homeomorphic to the m–cube I^m for some $0 \leq m \leq n$. Then

$$\widetilde{H}\left(\mathbb{S}^n \setminus X\right) = 0.$$

Proof. We will prove it by induction on m. If $m = 0$ then X is a point and $\mathbb{S}^n \setminus X \cong \mathbb{R}^n$. Assume $m \geq 1$ and that the result holds for $m-1$. Let $f : X \longrightarrow I^m$ be a homeomorphism.

Set

$$I^m_+ = \left\{(x_1, ..., x_m) \in I^m : x_1 \geq \frac{1}{2}\right\}$$
$$I^m_- = \left\{(x_1, ..., x_m) \in I^m : x_1 \leq \frac{1}{2}\right\}.$$

Note that

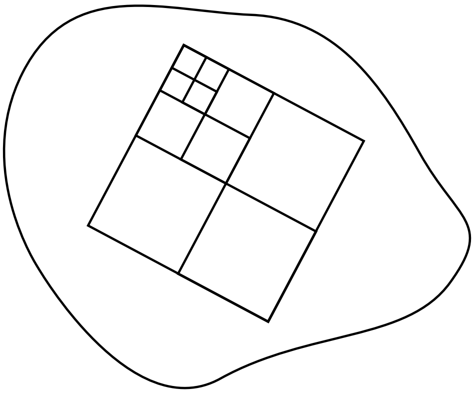
$$I^m_+ \cup I^m_- = I^m, \quad I^m_+ \cap I^m_- \cong I^{m-1}.$$

Now let $X^{\pm} = f^{-1}(I^m_{\pm})$ and $Y = f^{-1}(I^m_+ \cap I^m_-)$. We can apply the Mayer–Vietoris theorem to compute the homology of the space $\mathbb{S}^n \setminus Y = (\mathbb{S}^m \setminus X^+) \cup (\mathbb{S}^n \setminus X^-)$.

$$\begin{tikzcd}
\widetilde{H}_{i+1}(\mathbb{S}^n \setminus Y) \arrow[r] & \widetilde{H}_i(\mathbb{S}^n \setminus X) \arrow[r] & \widetilde{H}_i(\mathbb{S}^n \setminus x^+) \oplus \widetilde{H}_i(\mathbb{S}^n \setminus X^-) \arrow[r] & \widetilde{H}_i(\mathbb{S}^n \setminus Y)
\end{tikzcd}$$

By induction, $\widetilde{H}_{i+1}(\mathbb{S}^n \setminus Y) \cong \widetilde{H}_i(\mathbb{S}^n \setminus Y) \cong 0$. Assume for contradiction that there exists i and $\langle c \rangle \neq 0$ in $\widetilde{H}_i(\mathbb{S}^n \setminus X)$. Without loss of generality, we may assume that $\langle c_1 \rangle = H_i(j^+) \langle c \rangle \neq 0$ in $\widetilde{H}_i(\mathbb{S}^n \setminus X^+)$ where $j^+ : \mathbb{S}^n \setminus X \hookrightarrow \mathbb{S}^n \setminus X^+$.

By repeating this argument, we find a descending collection $\supset X_1 \supset X_2 \supset ...$ of closed sets homeomorphic to I^m together with non–zero homology classes $\langle c_k \rangle \in \widetilde{H}_i(\mathbb{S}^n \setminus X_k)$ such that the inclusion $j_k : \mathbb{S}^n \setminus X_{k-1} \hookrightarrow \mathbb{S}^n \setminus X_k$ sends $\langle x_{k-1} \rangle$ to $\langle c_k \rangle$.



Set $Z_k = \mathbb{S}^n \setminus X_k$. Writing $Z = \bigcap_k X_k$ Then Z_k is an increasing union of open sets such that $\mathbb{S}^n \setminus Z = \bigcup_k Z_k$. Thus by case (i) of theorem 7.10, we have that $\widetilde{H}_\bullet(\mathbb{S}^n \setminus Z) = \underrightarrow{\operatorname{colim}}\, \widetilde{H}_\bullet(Z_k)$. Since $\langle c_k \rangle$ maps to $\langle c_{k+1} \rangle$ by definition of the colimit the $\langle c_k \rangle$'s deine a non–zero element $\langle c_\infty \rangle \in \widetilde{H}_\bullet\, (\mathbb{S}^n \setminus Z)$. Z is homeomorphic to I^{m+1} so this contradicts the inductive hypothesis. □

Corollary 7.14. Let $S \subset \mathbb{S}^n$ be a subspace homeomorphic to $\mathbb{S}^m$ for some $m \leq n - 1$. Then

$$\widetilde{H}_1\,(\mathbb{S}^n \setminus \mathbb{S}) = \begin{cases} \mathbb{Z} & i = n - m - 1 \\ 0 & \text{otherwise} \end{cases}$$

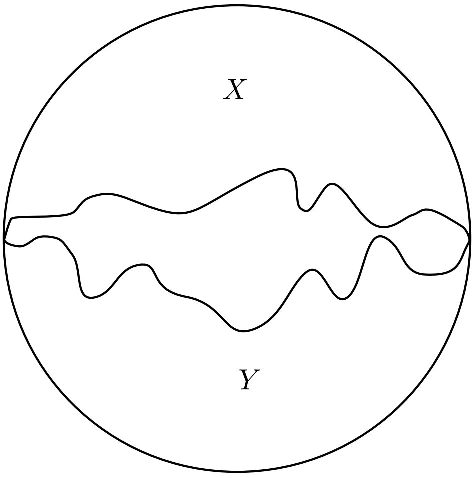
Proof. Again, the proof is carried out by induction on m. For $m = 0$, S consists of two points, and so $\mathbb{S}^n \setminus S \cong \mathbb{S}^{n-1}$ which has the desired homology. For the inductive step, write $S = B^+ \cup B^-$ corresponding to hemispheres. Let $Y = B^+ \cap B^- \cong \mathbb{S}^{m-1}$. Apply Mayer–Vietoris to get

$$\begin{tikzcd}
\widetilde{H}_i(\mathbb{S}^n \setminus B^+) \oplus \widetilde{H}_i(\mathbb{S}^n \setminus B^-) \arrow[d] & \widetilde{H}_{i-1}(\mathbb{S}^n \setminus B^+) \oplus \widetilde{H}_{i-1}(\mathbb{S}^n \setminus B^-) \\
\widetilde{H}_i(\mathbb{S}^n \setminus Y) \arrow[r] & \widetilde{H}_{i-1}(\mathbb{S}^n \setminus S) \arrow[u]
\end{tikzcd}$$

and $\widetilde{H}_i(\mathbb{S}^n \setminus B^+) \cong \widetilde{H}_i(\mathbb{S}^n \setminus B^-) \cong 0$ for all i. Since B^* are homeomorphic to cubes and $Y \cong \mathbb{S}^{m-1}$, the inductive hypothesis finishes the argument. ☐

Theorem 7.15 (*Jordan–Brouwer separation theorem*). If $f : \mathbb{S}^{n-1} \longrightarrow sphere^n$ is an embedding, then $\mathbb{S}^n \setminus f(\mathbb{S}^{n-1})$ has two components and $f(\mathbb{S}^{n-1})$ is the boundary of both of them.

Proof. Note that f is automatically a closed embedding. Let $S = f(\mathbb{S}^{n-1})$. Then by the previous result we have $\widetilde{H}_0(\mathbb{S}^n \setminus S) \cong \mathbb{Z}$ and hence $H(\mathbb{S}^n \setminus S) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus $\mathbb{S}^n \setminus S$ has two path components and since f is an embedding, path components coincide with connected components, call them X and Y. Let $\partial X = \overline{X} \setminus \operatorname{Int} X$, we want to show that $\partial X = S$. Since $X \cup S$ is closed we have $\partial X \subset S$ and so we need only show $S \subset \partial X$, and this is given by point set topology. ☐



Theorem 7.16 (*Brouwer's invariance of domain*). Suppose U and U' are two subsets of $\mathbb{S}^n$ and $f : U \longrightarrow U'$ is a homeomorphism. If U is open then so is U'.

8 Cellular homology

We will now take a look at another construction of homology which will make computations easier. Singular homology has the disadvantage that $C_\bullet^{\text{sing}}(X)$ can be overwhelmingly big, it can be of uncountable rank when X is not dsirable. In order to compute singular homology, one needs to use tricks such as colimits and the Mayer–Vietoris theorem. Cellular homology produces $C_\bullet^{\text{cell}}(X)$ of finite rank if X is compact. It is easier to compute, but the disadvantage to cellular homology is that it can only be defined for cell complexes (CW complexes) and not for every topological space. However, we will see that this is not a tragic disadvantage.

Historically, topological spaces were understood as spaces which could be triangulated, that is, divided into simplices. A cell complex will be the modern version of this, and we will see that cell complexes are spaces wich can be decomposed as a set of cells.

Definition 8.1. A map $f : X \longrightarrow Y$ is said to be a weak homotopy equivalence if $\pi_n(f) : \pi_n(X) \longrightarrow \pi_n(Y)$ is an isomorphism for all n.

Notice that a homotopy equivalence is in particular a weak homotopy equivalence but the converse is not true. Also, if $f : X \longrightarrow Y$ is a weak homotopy equivalence, then $H_n(f) : H_n(X) \longrightarrow H_n(Y)$ is an isomorphism for every n. So homology cannot distinguish between weakly homotopy equivalent spaces, they are the same as far as homology is concerned.

Theorem 8.2. If X is any toplogical space, then there exists a cell complex Y and a weak homotopy equivalence $f : X \longrightarrow Y$.

This theorem tells us that the disadvantage that we talked about earlier is not that important since we can always find a cell complex with the same homology as our original space.

Theorem 8.3. If X and Y are connected cell complexes, then a weak homology equivalence is a genuine homotopy equivalence.

These two theorems motivate the use of cellular complexes since they show that their disadvantages are not too important, and they are easier to work with.

Definition 8.4. Let X' be a closed subspace of X and suppose $f : X' \longrightarrow Y$ is continuous. The adjunction space $X \cup_f Y$ is the sapce $X \sqcup Y / \sim$ where $\sim$ is the smallest equivalence relation such that $x \sim f(x)$ for all $x \in X'$.

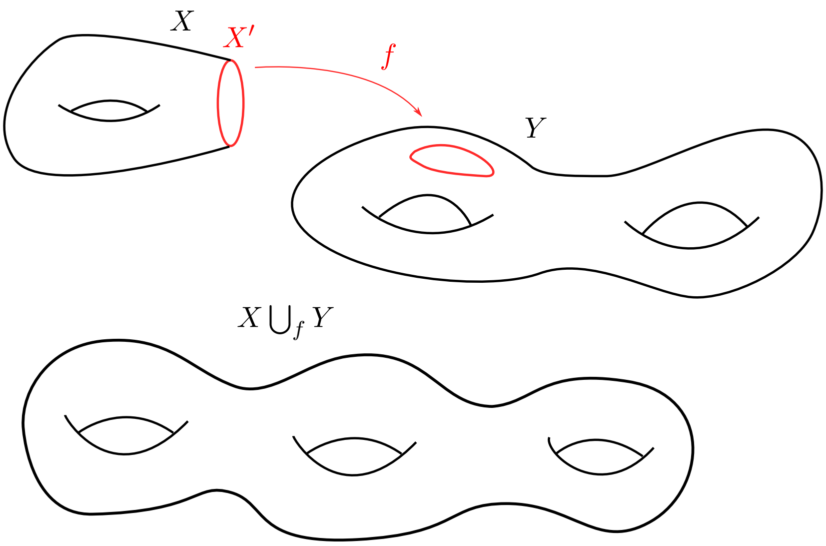


Figure 8.1: Example of the adjunction space of X and Y when X is a torus, Y is a torus with two holes and X' is the boundary of a ball in X.

A adjunction space $X \cup_f Y$ or, when f is clear, $X \cup_{X'} Y$, is the space obtained by gluing Y to X along X'. We introduce the following notation.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend right, "g"'] & X \sqcup Y \arrow[r] & X \sqcup Y/\sim\, = X \cup_f Y \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[rr, bend right, "j"'] & X \sqcup Y \arrow[r] & X \sqcup Y/\sim\, = X \cup_f Y \end{tikzcd}$$

Lemma 8.5. Adjunction spaces are pushouts in Top.

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, hook, "i"'] & Y \\ X & \end{tikzcd} \quad \text{has pushout} \quad \begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, hook, "i"'] & Y \arrow[d, "j"] \\ X \arrow[r, "g"] & X \cup_f Y \end{tikzcd}$$

and the following point set topological properties hold.

(i) j is a closed embedding.

(ii) $g|_{X \setminus X'}$ is an open embedding.

(iii) The quotient map $X \sqcup Y \longrightarrow X \cup_f Y$ is closed if and only if f is closed.

(iv) If X, Y are Hausdorff and X' is compact, then $X \cup_f U$ is Hausdorff.

(v) If X, Y are T_1, then so is $X \cup_f Y$.

(vi) If x, Y are weakly Hausdorff, then so is $X \cup_f Y$.

(vii) If X is compact and $X \cup_f Y$ is Hausdorff, then $g : X \longrightarrow g(X)$ is a quotient map.

Remark. Pushouts do not necessarily preserve the category of Hausdorff spaces (a subcate-

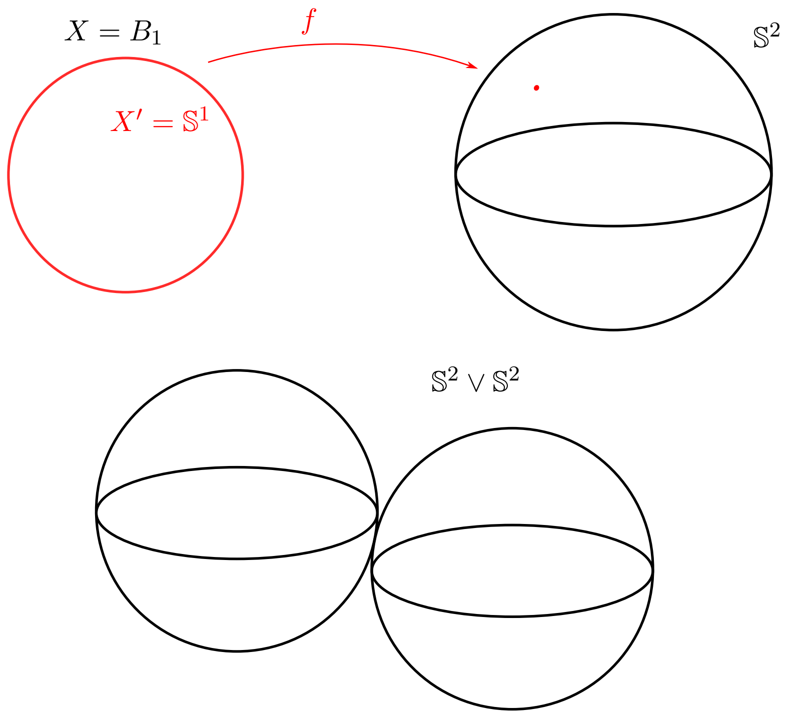


Figure 8.2: Example of the adjunction space of X and Y when X is the unit ball, X' is its boundary, and Y is $\mathbb{S}^2$.

gory of Top. However, T_1 or weakly Hausdorff are preserved, as the lemma suggests.

Definition 8.6. A coembedding is a map which is a quotient map onto its image.

Here is a "converse" to the preceding lemma. Recall that an embedding is a map that is a homeomorphism into its image.

Lemma 8.7. Suppose the following diagram commutes.

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "i"'] & Y \arrow[d, "j"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

Assume in addition that

(i) i and j are closed embeddings.

(ii) g induces a bijection $X \setminus i(X') \longrightarrow Z \setminus j(Y)$.

(iii) $g(X)$ is a closed coembedding.

Then Z is homeomorphic to the induced adjunction space $X \cup_{i(X')} Y$ and hence it is a pushout.

Proof. We already know that pushouts are unique and that the adjunction space is a pushout. We need only verify the universal property

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "i"'] & Y \arrow[d, "j"] \arrow[ddr, bend left, "k"] & \\
X \arrow[r, "g"] \arrow[drr, bend right, "h"'] & Z \arrow[dr, dashed, "u"] & \\
& & W
\end{tikzcd}$$

and prove that there exists a unique continuous map u. It is clear that Z is a pushout in Sets. Thus, there exists a unique map $u : Z \longrightarrow W$ by the pushout property in Sets and hence we need only check u is continuous. However, since j is an embedding, $u|_{j(Y)}$ is continuous, and by the same argument with g, $u|_{g(X)}$ is continuous. Therefore u is continuous. □

We now introduce some notation.

Definition 8.8. Define $E^n = \text{Int}\, B^n = B^n \setminus \mathbb{S}^{n-1}$. We call E^n the standard n–cell. If X is a topological space and $E \subset X$ is homeomorphic to E^n, then we say E is an n–cell in X.

We say X is obtained from Y by adding an n–cell if X is the adjunction space $X = B^n \cup_{\mathbb{S}^{n-1}} Y$. In general, we say X is obtained from Y by adding n–cells if X is obtained from Y by adding (possibly uncountably many) n–cells. Roughly speaking, a cell complex is something built by adding cells.

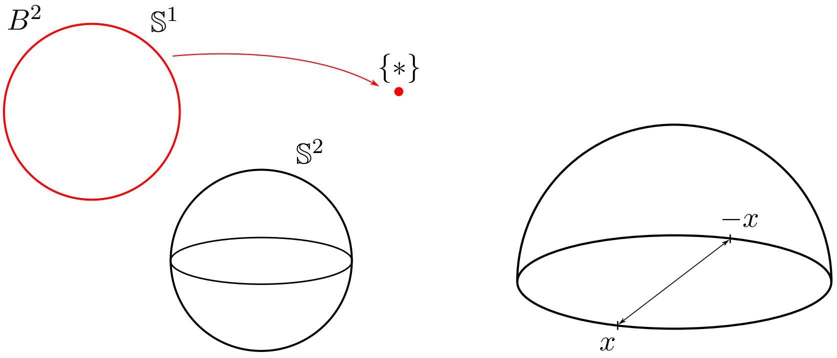


Figure 8.3: On the left, $\mathbb{S}^2$ as $\{*\}$ after adding a 2–cell. On the right, the identification in (iv).

Example

(i) $\mathbb{S}^n$ is obtained from $\{*\}$ by adding an n–cell.

(ii) $\mathbb{S}^m \times \mathbb{S}^n$ is obtained from $\mathbb{S}^m \vee \mathbb{S}^n$ by adding an $(n+m)$–cell (see figure 8.3).

(iii) $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is obtained from the lemniscate by adding a 2–cell.

(iv) Consider $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$ where $x \sim y$ if and only if $x = ty$ for some $t \neq 0$. This is also $\mathbb{S}^n / \sim$ where $x \sim -x$ (figure 8.3). This is the same thing as taking an upper hemisphere and identifying points on the opposite sides of the equator.
This shows that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n–cell. Thus by induction, $\mathbb{R}P^n$ is obtained by succesively adding n–cells, $\mathbb{R}P^n = E^0 \cup ... \cup E^n$. This means start with $E^0 = \{*\}$ (by definition) and attach a 1–cell, then a 2–cell, etc. Similarly, $\mathbb{C}P^n = E^0 \cup E^2 \cup ... \cup E^{2n}$. Here the union denotes "attaching",

not the union of sets. This will show that

$$C_i^{\text{cell}}(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & i = 0, ..., n \\ 0 & \text{otherwise} \end{cases}, \quad C_i^{\text{cell}}(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & i = 0, ..., 2n \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8.9. A space X is said to be a cell complex if we find subspaces $X^i \subset X$ such that

(i) X^{i+1} is obtained from X^i by adding $(i+1)$–cells and

$$\emptyset =: X^{-1} \subset X^0 \subset ... \subset X.$$

(ii) X carries the colimit topology of the X^i's.

We will call X^i the ith skeleton of X.

Remark. The $X^{`\prime}$s are not necessarily unique, if there exists N such that $X^N = X^{N+1} = ... = X$, i.e. there are no cells of "dimension" greater than N, then we say X is an N–dimensional cell complex.

$\mathbb{R}P^n$ is an n–dimensional cell complex, and $\mathbb{C}P^n$ is a $2n$–dimensional cell complex.

Definition 8.10. If $X' \subset X$ is any subspace we say (X, X') is a relative cell complex if there exist X^i's such that $X' = X^{-1} \subset X^0 \subset ... \subset X$ where X^{i+1} is obtained from X^i by adding i–cells and X carries the colimit topology of the X^i's.

Theorem 8.11. Let X be a cell complex with skeleton $\{X^i\}$. Then

(i) $X^n \hookrightarrow X$ is a closed embedding.

(ii) X is Hausdorff.

(iii) $H_\bullet(X) = \varinjlim_n H_\bullet(X^n)$.

Proof. Notice that (i) and (ii) follow from the point set topology lemma from before and (iii) is the same as theorem 7.9. ☐

In order to exploit (iii), we need to understand the relation between $H_\bullet(X = B^n \cup_{\mathbb{S}^{n-1}} Y)$ and $H_\bullet(Y)$, i.e., how homology changes when attaching an n–cell. Let us take a look at an example.

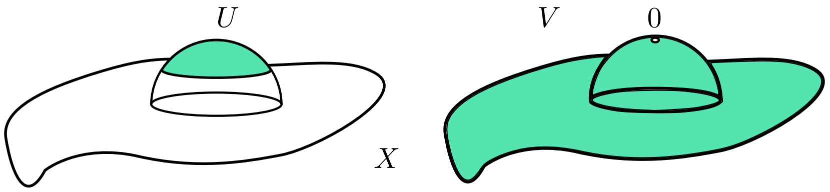
Proposition 8.12. Suppose $f : \mathbb{S}^{n-1} \longrightarrow Y$ is continuous with $n \geq 2$. Then there is a long exact sequence in homology. Set $X = B^n \cup_f Y$ and denote by $j : Y \longrightarrow X$ the map induced by the inclusion $Y \hookrightarrow B^n \sqcup Y$. The long exact sequence

$$... \longrightarrow H_k(\mathbb{S}^{n-1}) \xrightarrow{H_k(f)} H_k(Y) \xrightarrow{H_k(j)} H_k(X) \longrightarrow H_{k-1}(\mathbb{S}^{n-1}) \longrightarrow ...$$

ends with

$$H_0(\mathbb{S}^{n-1}) \longrightarrow \mathbb{Z} \oplus H_0(Y) \longrightarrow H_0(X) \longrightarrow 0$$

Proof. Write $X = U \cup V$, where U is the image through f of the open ball of radius $1/2$ and V is everything apart from $0 \in B^n$.



Then $U \cap V$ is homotopy equivalent to $\mathbb{S}^{n-1}$. Also, U is contractible and moreover V deformation retracts onto Y. Let $g : B^n \longrightarrow X$ be the induced map from $B^n \hookrightarrow B^n \sqcup Y$. Define $H : V \times I \longrightarrow V$ given by

$$H(x,t) = \begin{cases} x & x \in Y \\ g\left((1-t)z + \frac{tz}{|z|}\right) & x = g(z) \in E^n \setminus 0. \end{cases}$$

Plug in this to the Mayer–Vietoris to obtain the desired sequence

$$\dots \longrightarrow H_k(U \cap V) \xrightarrow{H_k(h)} H_k(U) \oplus H_k(V) \longrightarrow H_k(X) \longrightarrow H_{k-1}(U \cap V) \longrightarrow \dots$$

This gives the sequence as stated, apart from the fact that we need to identify the map $H_k(h)$ with $H_k(f)$. For $k > 0$ this is the inclusion $U \cap V \hookrightarrow V$ via h, and we need to show that this map is the same as $H_k(f)$.

$$\begin{tikzcd}
U \setminus 0 \arrow[r, "id"] \arrow[d, hook] & U \cap V \arrow[d, hook, "h"] \\
B^n \setminus 0 \arrow[r] & V \\
\mathbb{S}^{n-1} \arrow[r, "f"] \arrow[u, hook] & Y \arrow[u, hook]
\end{tikzcd}$$

The blue maps induce isomorphisms on homology. The top map, which is induced from g, also gives an isomorphism in homology. Thus modulo this isomorphisms, $H_k(f) = H_k(h)$. □

Corollary 8.13. Suppose $n \geq 2$. Then for $k \neq n-1$ or n,

$$H_k(Y) = H_k(X).$$

Proof. Use the long exact sequence. □

Recall that in theorem 6.18 we claimed (without proof) that under favorable circumstances

$$H_k(X, X') \cong \widetilde{H}_k(X/X').$$

We are now going to give the proof for this theorem, but first, let us say what “well behaved” means.

Definition 8.14. Let X' be a subspace of X. We say X' is a strong deformation retract if there exists a continuous map $r : X \longrightarrow X'$ such that

(i) $r \circ i = id_{X'}$, where $i : X' \hookrightarrow X$ is the inclusion.

(ii) $i \circ r \simeq id_X$ rel X', that is, there exists $H : X \times I \longrightarrow X$ such that

$$\begin{aligned} H(x,0) &= x \quad & x \in X \\ H(x,1) &\in X' \quad & x \in X \\ H(y,t) &= y \quad & y \in X', t \in I. \end{aligned}$$

Definition 8.15. Let $X' \subset X$. We say X' is a collapsible subspace if

(i) X' is closed in X.

(ii) There exists a neighborhood U of X' in X such that X' is a strong deformation retract of U.

For example, Y is a collapsible subspace of $B^n \cup_f Y$. We now re–state theorem 6.18 in terms of the definitions we just gave.

Theorem 8.16. Let $X' \subset X$ be collapsible and let $\rho : X \longrightarrow X/X'$ be the quotient map. Then

$$H_k(\rho) : H_k(X, X') \longrightarrow H_k(X/X', *)$$

is an isomorphism.

Proof. Let U be a neighborhood of X' in X such that X' is a strong deformation retract of U and let $j : X' \hookrightarrow U$.

$$\begin{tikzcd}
H_k(X') \arrow[r] \arrow[d, "H_k(j)"] & H_k(X) \arrow[r] \arrow[d, "id"] & H_k(X, X') \arrow[r] \arrow[d, "l"] & H_{k-1}(X') \arrow[r] \arrow[d, "H_{k-1}(j)"] & H_{k-1}(X) \arrow[d, "id"] \\
H_k(U) \arrow[r] & H_k(X) \arrow[r] & H_k(X, U) \arrow[r] & H_{k-1}(U) \arrow[r] & H_{k-1}(X)
\end{tikzcd}$$

and by the five lemma, l is an isomorphism. If we now call $\overline{X} := X \setminus X'$ and $\overline{U} := U \setminus X'$, a similar argument shows that $H_k(\overline{X}, *) \cong H_k(\overline{X}, \overline{U})$. Now

$$\begin{tikzcd}
 & H_k(X,U) & \\
H_k(X,X') \arrow[ur, "\cong"] \arrow[d, "\rho"'] & & H_k\left(X\setminus X', U\setminus X'\right) \arrow[ul, "\cong\ \text{(excision)}"'] \arrow[d, "H_k(\rho)"] \\
H_k(\overline{X},*) \arrow[dr, "\cong"'] & & H_k(\overline{X}\setminus *, \overline{U}\setminus *) \arrow[dl, "\cong"] \\
 & H_k(\overline{X},\overline{U}) &
\end{tikzcd}$$

where $H_k(\rho)$ is an isomorphism as $\rho|_{X\setminus X'}$ is a homeomorphism. □

Definition 8.17. A map $f : (X, X') \longrightarrow (Y, Y')$ is a relative homeomorphism if $f|_{X\setminus X'}$ is a homemoprhism from $X \setminus X'$ to $Y \setminus Y'$.

Now we state a generalization of the previous theorem.

Theorem 8.18. Let $X' \subset X$ and $Y' \subset Y$ be collapsible and $f : (X, X') \longrightarrow (Y, Y')$ a relative homeomorphism. Assume in addition that $\overline{f} : \overline{X} \longrightarrow \overline{Y}$ is a homeomorphism. Then

$$H_k(f) : H_k(X, X') \longrightarrow H_k(Y, Y')$$

is an isomorphism.

Remark.

(i) $\rho : (X, X') \longrightarrow (Y, Y')$ is a relative homeomorphism.

(ii) If X is compact and Y is compact Hausdorff, then $\overline{f}$ is a homeomorphism.

Proof.

$$\begin{tikzcd}
H_k(X,X') \arrow[r, "H_k(f)"] \arrow[d, "\cong"] & H_k(Y,Y') \arrow[d, "\cong"] \\
H_k(\overline{X},*) \arrow[r, "H_k(\overline{f})"] & H_k(\overline{Y},*)
\end{tikzcd}$$

□

Y is a collapsible subspace of $B^n \cup_f Y$.

Definition 8.19. Let X be a cell complex and $X' \subset X$ a subspace. We say that X' is a subcomplex of X if for any cell E of X, $E \cap X' \neq \emptyset \implies \overline{E} \subset X'$. This implies that X' is a cell complex and (X, X') is a relative cell complex.

To apply the tools we have developed until now, we will now show that subcomplexes are collapsible. The proof for this result is non–examinable since it is a bit tedious.

Proposition 8.20. Let X' be a subcomplex of X. Then X' is collapsible.

Sketch of proof. For each cell E_λ of X which is not in X', choose a point x_λ, for example x_λ could correspond to $0 \in E^n_\lambda$. Now define

$$Y^n := \left\{ x_\lambda : \ E_\lambda \text{ is an } n\text{–cell of } X \text{ not in } X' \right\}.$$

Regard (X, X') as a relative cell complex with skeleton

$$X' = X^{-1} \subset X^0 \subset \ldots \subset X.$$

X^0 is obtained from X' by adding 0–cells, X^1 is obtained by X^0 by adding i–cells, and so on. Now we claim that X^{n-1} is a strong deformation retract of $X^n \setminus Y^n$. The proof for this claim is by arguing that this is the same as the proof that Y was a strong deformation retract of V in proposition 8.12. Assuming the claim, we have retractions

$$r_n : X^n \setminus Y^n \longrightarrow X^{n-1},$$

and to build a retraction from a neighborhood of X' in X to X', one composes the r_n's and checks that everything is continuous. ☐

Corollary 8.21 (*Excision with subcomplexes*)**.** Suppose X is a cell complex and X' and X'' are subcomplexes such that $X = X' \cup X''$. Then the inclusion $(X'', X' \cap X'') \hookrightarrow (X, X')$ induces isomorphisms on homology.

Proof. The quotient space $X''/X' \cap X''$ is homeomorphic to X/X' since both are obtained from the cells in X'' that are not in X'. Since the spaces are collapsible, the result follows from the previous theorem. ☐

Corollary 8.22 (*Mayer–Vietoris for subcomplexes*)**.** Let X be a cell complex with subcomplexes X' and X'' such that $X = X' \cup X''$. Then there exists a long exact sequence

$$\ldots \longrightarrow H_k(X' \cap X'') \xrightarrow{(i', i'')} H_k(X') \oplus H_k(X'') \xrightarrow{j' - j''} H_k(X) \longrightarrow H_{k-1}(X' \cap X'') \longrightarrow \ldots$$